ICMRS

2023

6TH INTERNATIONAL CONFERENCE
ON MATHEMATICAL AND RELATED SCIENCES

NOVEMBER 20-22, 2023

6TH INTERNATIONAL CONFERENCE ON MATHEMATICAL AND RELATED SCIENCES

BOOK OF PROCEEDINGS

ISBN: 978-785-70564-8-1

2023

6TH INTERNATIONAL CONFERENCE ON MATHEMATICAL AND RELATED SCIENCES

ICMRS 2023

NOVEMBER 20-22, 2023

EDITORS

ERHAN SET, PhD

AHMET OCAK AKDEMİR, PhD

MEHMET ZEKİ SARIKAYA, PHD

ALPER EKİNCİ, PhD

All papers have been peer reviewed.

CHAIR

Prof. Dr. Ahmet Ocak AKDEMİR
Ağrı İbrahim Çeçen University *Türkiye*

ORGANIZING COMMITTEE

Prof. Dr. Erhan SET
Ordu University *Türkiye*

Prof. Dr. Mehmet Zeki SARIKAYA
Düzce University *Türkiye*

Assoc. Prof. Dr. Alper EKİNCİ
Bandırma Onyedi Eylül University *Türkiye*

SECRETARY

Dr. Barış ÇELİK
Ordu University *Türkiye*

KEYNOTE SPEAKERS

Prof. Dr. Miodrag SPALEVIC
University of Belgrade-SERBIA

Prof. Dr. Leo G. REBHOLZ
Clemson University - USA

Prof. Dr. Luca GRILLI
Università degli Studi di Foggia-ITALY

Prof. Dr. Jehad Jumah AL JARADEN
Al Hussein Bin Talal University – JORDAN

Prof. Dr. Zakia HAMMOUCH
ENS Moulay Ismail University – MOROCCO

Prof. Dr. Ali SHOKRI
University of Maragheh - IRAN

SCIENTIFIC COMMITTEE

Abdallah BENAISSA
University of Batna
Algeria

Abdullah ÇAĞMAN
Ağrı İbrahim Çeçen University
Türkiye

Abdullah ERGÜN
Sivas Cumhuriyet University
Türkiye

Alexandar Pejcev
University of Belgrade
Serbia

Ali AKGÜL
Siirt University
Türkiye

Ali ÇAKMAK
Ağrı İbrahim Çeçen University
Türkiye

Ali SHOKRI
University of Maragheh
Iran

Allal GUESSAB
Université de Pau et des Pays de I'Adour
France

Anthony SOFO
Victoria University
Australia

Arif RAFIQ
Hajvery University
Pakistan

Aytül GÖKÇE
Ordu University
Türkiye

Bayram ŞAHİN
Ege University
Türkiye

Borhan ALBISS
Jordan University of Science and Technology
Jordan

Burhaneddin İZGİ
İstanbul Technical University
Türkiye

Canan ÇİFTÇİ
Ordu University
Türkiye

Cetin YILDIZ
Atatürk University
Türkiye

Davorka JANDRLIC
University of Belgrade
Serbia

Djamila SEBA
University of Boumerdes
Algeria

Donal O REGAN
National University of Ireland
Ireland

Dusan DJUKIC
University of Belgrade
Serbia

Ekrem KADIOĞLU
Atatürk University
Türkiye

Elnara MAMMADOVA
Baku State University
Azerbaijan

Elvan AKIN
Missouri University of Science and Technology
USA

Emrah Evren KARA
Düzce University
Türkiye

CONTENTS

Frequency Analysis of Moderately Thick Composite Panels with Negative Gaussian Curvature

İlke ALGÜL[1] and Ahmet Sinan ÖKTEM[1]

[1]*Department of Mechanical Engineering,*
Gebze Technical University, 41400, Gebze-TURKEY

ilkees@gtu.edu.tr
sinan.oktem@gtu.edu.tr

ABSTRACT

This study explores the frequency analysis of moderately thick laminated cylindrical, spherical and negative Gaussian curvature panel for various combinations of boundary conditions. The analysis incorporates a first-order shear deformation theory along with an extension of linear strain-displacement relationships to account for transverse shear effects throughout the thickness direction. The equilibrium equations for laminated composite shells are derived using the virtual work principle. Generalized differential quadrature method is employed here to solve the free vibration of plates and panels shells made up from symmetric and antisymmetric cross-ply laminated composites. Utilizing the GDQ method in the governing differential equation transforms the problem into a generalized eigenvalue problem, leading to the determination of the frequency parameter. Natural frequencies are explained with tabular data by investigating the effect of various parameters such as effects of stacking lamination, boundary conditions and the effect of curvature.

1. INTRODUCTION

Curved composite panel with uniform thickness h, length a, and width b is given in Figure 1.1. The x, y and z stated the orthogonal curvilinear coordinate system attached to the middle surface of the shell (z=0). R_1 and R_2 are denoted the principal radius of the middle surface of the panel curvature.

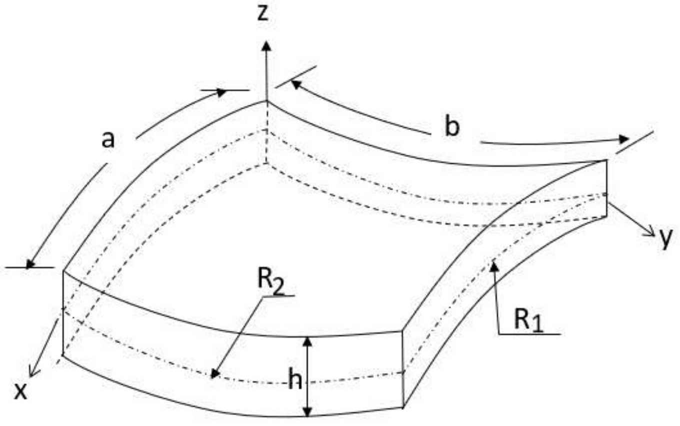


Figure 1.1. Negative Gaussian Curvature Panel (R_1=R, R_2=-R)

The displacement field at general point (*x*, *y*, and *z*) of the panel based on first-order shear deformation theory may be written as:

$$u(x, y, z, t) = u_0(x, y, t) + z.\theta_x(x, y, t) \tag{1.1a}$$

$$v(x, y, z, t) = u_0(x, y, t) + z.\theta_y(x, y, t) \tag{1.1b}$$

$$w(x, y, z, t) = w_0(x, y, t) \tag{1.1c}$$

where u_0, v_0, w_0 are the displacement field of a point on the middle surface of the shell along the *x*, *y* and *z* axes. θ_x and θ_y are the rotations around the *y* and *x* axes, respectively.

The strain-displacement relations for the curved panels using the displacement fields in equations. (1.1a-c) from the theory of elasticity in curvilinear coordinates are given below for small elastic deformation.

$$\varepsilon_x = \frac{\partial u_0}{\partial x} + z\frac{\partial \theta_x}{\partial x} + \frac{w_0}{R_x} \tag{1.2a}$$

$$\varepsilon_y = \frac{\partial v_0}{\partial y} + z\frac{\partial \theta_y}{\partial y} + \frac{w_0}{R_y} \tag{1.2b}$$

$$\gamma_{xy} = \frac{\partial v_0}{\partial x} + \frac{\partial u_0}{\partial y} + z\frac{\partial \theta_x}{\partial y} + z\frac{\partial \theta_y}{\partial x} + \frac{1}{2}\left(\frac{1}{R_y} - \frac{1}{R_x}\right)\left(\frac{\partial v_0}{\partial x} - \frac{\partial u_0}{\partial y}\right) \tag{1.2c}$$

$$\gamma_{yz} = \theta_y + \frac{\partial w_0}{\partial y} - \frac{v_0}{R_y} \tag{1.2d}$$

$$\gamma_{xz} = \theta_x + \frac{\partial w_0}{\partial x} - \frac{u_0}{R_x} \tag{1.2e}$$

For the sake of brevity, the derivation of equilibrium equations of the composite doubly curved panel using the virtual work principle is not explained here. Further explanations are given in [1-2]. With the use of the virtual work principle, five coupled fourth-order linear governing differential equations of a moderately thick laminated composite doubly curved panel are as given below:

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} + \frac{Q_{xz}}{R_x} - \frac{\partial M_{xy}}{\partial y}\left(\frac{1}{2}\left(\frac{1}{R_y} - \frac{1}{R_x}\right)\right) = I_0\frac{\partial u_0}{\partial t^2} + I_1\frac{\partial^2 \theta_x}{\partial t^2} \tag{1.3a}$$

$$\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} + \frac{Q_{yz}}{R_y} + \frac{\partial M_{xy}}{\partial x}\left(\frac{1}{2}\left(\frac{1}{R_y} - \frac{1}{R_x}\right)\right) = I_0\frac{\partial^2 v_0}{\partial t^2} + I_1\frac{\partial^2 \theta y}{\partial t^2} \tag{1.3b}$$

$$-\frac{N_x}{Rx} - \frac{N_y}{Ry} + \frac{\partial Q_{yz}}{\partial y} + \frac{\partial Q_{xz}}{\partial x} = I_0\frac{\partial^2 w_0}{\partial t^2} + q(x, y, t) \tag{1.3c}$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_{xz} = I_1\frac{\partial^2 u_0}{\partial t^2} + I_2\frac{\partial^2 \theta x}{\partial t^2} \tag{1.3d}$$

$$\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_{yz} = I_1\frac{\partial^2 v_0}{\partial t^2} + I_2\frac{\partial^2 \theta y}{\partial t^2} \tag{1.3e}$$

The explanation of the in-plane and transverse stress and moment resultants (N_x, N_y, N_{xy}, $M_x, M_y, M_{xy}, Q_{xz}, Q_{yz}$) for a laminated panel can be found in [1]. In equations (1.3a-e), laminate mass inertia terms are expressed as:

$$(I_0, I_1, I_2) = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \rho^k (1, z, z^2)\, d_z. \tag{1.4}$$

where ρ(k) indicates the density of the k-th layer and $q(x, y, t)$ is the distributed lateral load at the top of the laminate given in Eq. (1.3c).

The problem is solved for the boundary conditions below explained mathematically:

Simply supported type 3 (SS3) boundary conditions identified at the edges,

$$x = 0, a: u_2 = u_3 = \Theta_2 = M_1 = N_1 = 0 \tag{1.5a}$$
$$y = 0, b: u_1 = u_3 = \Theta_1 = M_2 = N_2 = 0 \tag{1.5b}$$

Mixed type simply supported type (SS1-SS2-SS3-SS4) boundary conditions identified at the edges,

$$(SS2)\ x = 0:\ u_1 = u_3 = \Theta_2 = M_1 = N_6 = 0 \tag{1.6a}$$
$$(SS1) x = a:\ u_3 = \Theta_2 = M_1 = N_1 = N_6 = 0 \tag{1.6b}$$
$$(SS4) y = 0:\ u_1 = u_2 = u_3 = \Theta_1 = M_2 = 0 \tag{1.6c}$$
$$(SS3) y = b:\ u_1 = u_3 = \Theta_1 = M_2 = N_2 = 0 \tag{1.6d}$$

SS3-RS4 boundary conditions defined in [4] identified at the edges,

$$(RS4) x = 0, a: u_1 = u_2 = \Theta_2 = M_1 = Q_1 = 0 \tag{1.7a}$$
$$(SS3) y = 0, b: u_1 = u_3 = \Theta_1 = M_2 = N_2 = 0 \tag{1.7b}$$

2. STATEMENT OF PROBLEM

GDQ method [4] is employed to obtain numerical solution in order to solve the governing equation with the proposed boundary conditions. The GDQ provides a means to approximate derivatives of a smooth function, where the method evaluates the derivative of a given order at a specific point by combining the function values from all domain points. The proposed mathematical formulation starts with the displacement functions which involves the time variable (t) as well as position variables (x, y). Displacement functions can be written as follows:

$$u^{(T)}(x, y, t) = U^T(x, y) e^{i\omega t} \tag{2.1}$$

where $\boldsymbol{U}^{(\boldsymbol{T})}$ is composed of function of spatial variables and is written in matrix form as:

$$\boldsymbol{U}^{(\boldsymbol{T})} = [U_1(x,y) \quad U_2(x,y) \quad U_3(x,y) \quad \Theta_1(x,y) \quad \Theta_2(x,y)]^T \tag{2.2}$$

$U_1, U_2, U_3, \Theta_1, \Theta_2$ can be expressed in terms of nodal displacement components and Lagrange polynomials.

$$U_i(x, y) = \sum_{m=1}^{M} \sum_{n=1}^{N} L_m(x) S_n(y)\, u_i(x, y)\ (i=1, 2, 3) \tag{2.3a}$$

$$\Theta_i(x,y) = \sum_{m=1}^{M}\sum_{n=1}^{N} L_m(x)S_n(y)\,\Theta_i(x,y)\ (i=1,2) \quad (2.3b)$$

$u_i(x_i, y_j)$ are field variables at grid point $(i\ ,\ j)$ and the first and 2nd derivatives of a two-dimensional field variable with respect to x or y is mentioned in [3]. Global assembling of governing equations and boundary conditions with the substitution of field variables (displacement functions) results in the matrix form as:

$$M\ddot{U} + KU = 0 \quad (2.4)$$

where K is the stiffness matrix, [m] is the mass matrix, {U} and $\{\ddot{U}\}$ are the nodal displacement and acceleration vector, respectively. Written matrix in the way causes numerical instabilities and ill-conditioned matrices [5]. Thus, it is required to subdivide the matrix as boundary equations b and domain equations d. It is possible to write:

$$\begin{bmatrix} [K_{bb}] & [K_{bd}] \\ [K_{db}] & [K_{dd}] \end{bmatrix} \begin{Bmatrix} \Delta_b \\ \Delta_d \end{Bmatrix} - \omega^2 \begin{bmatrix} [0] & [0] \\ [0] & [M_{dd}] \end{bmatrix} \begin{Bmatrix} \Delta_b \\ \Delta_d \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (2.5)$$

It is possible to rewrite:

$$\left(K_{dd} - K_{db}\left({K_{bb}}^{-1}\right)K_{bd}\right)\Delta_d = \omega^2 M_{dd}\Delta_d \quad (2.6a)$$

$$(\hat{K}\text{-}\omega^2\hat{M})\ \Delta_d = 0 \quad (2.6b)$$

The natural frequencies of the structure, represented by ω can be found by solving the standard eigenvalue problem in equation (2.6b). To achieve this, the MATLAB software's built-in eigs function is utilized, which yields the results in terms of the natural frequencies for the structures under investigation.

3. NUMERICAL RESULTS AND DISCUSSION

In this subsection, numerical examples for free vibration analysis of moderately thick plate and panels are presented by investigating the effect of various parameters such as effects of stacking lamination, material types, the effect of curvature, loading conditions, etc. The following dimension mechanical properties are assumed for validation of composite cross-ply laminate plate with [6]:

Material Type 1: $E_1/E_2 = 40$, $G_{12} = G_{13} = 0.6E_2$, $G_{23} = 0.5E_2$, $\nu_{12} = 0.25$ and $\nu_{21} = 0.00625$.

The non-dimensional frequency parameter, λ specified as

$$\lambda = \frac{\omega b^2}{\pi^2}\sqrt{\frac{\rho h}{D_0}} \quad (3.1)$$

where

$$D_0 = \frac{E_2 h^3}{12(1 - \nu_{12}\nu_{21})} \quad (3.2)$$

Table 6.1 presents the frequency parameters from the first validation along with the values from Liew [6]. The frequency results obtained by Srinivas et al. [7] and Liew et al. [6] were based on 3D elasticity theory. It is observed that at a/h=100, the frequency parameters from the 3D elasticity theory are in close agreement with the current results. However, at a higher thickness ratio (a/h=10), the first order shear deformation theory (FSDT) used in the proposed frequency model tends to provide lower eigenvalues after the third mode sequence number compared to the 3D elasticity theory. Notably, the error between present results and shear deformation theories results becomes more significant for higher modes for a/h=10.The difference is ascribed to FSDT including a shear correction factor, thickness shear deformation, and rotary inertia, which results in a reduced estimation of plate stiffness.

Table 6.2 validates the response of frequency parameters for the simply supported cross-ply [0°/90°/90°/0°] laminated plate at different length-to-thickness ratios (a/h). The comparison of our results with those of Reddy and Phan [8], who used a high order shear deformation theory, demonstrates good agreement. Notably, the fundamental frequencies in Table 6.2 show a significant impact from varying thickness, with a general trend of increasing fundamental frequency as the a/h ratio decreases.

Table 6.1: Comparison of frequency parameter, λ for a simply supported square plate. (v=0.3, a/h= 10,100).

a/h	Results	Mode sequence number					
		1	2	3	4	5	6
100	Srinivas et al. [7]	1.999	4.995	4.995	7.988	9.981	9.981
	Liew [6]	2.000	5.000	5.000	8.000	9.999	9.999
	Present Study	1.9993	5.0024	5.0024	7.9978	10.0279	10.0279
10	Srinivas et al. [7]	1.9342	4.6222	4.6222	7.1030	8.6618	8.6618
	Liew [6]	1.931	4.605	4.605	7.064	8.605	8.605
	Present Study	1.9317	4.6109	4.6109	6.5234	6.5234	7.0747

Table 6.2: Comparison of frequency parameter, λ of a simply supported cross-ply [0°/90°/90°/0°] laminated plate.

a/h	0.01	0.02	0.04	0.05	0.08	0.10	0.20	0.25	0.50
Reddy and Phan [8]	6.578	6.475	6.330	6.196	5.708	5.355	3.854	3.331	1.956
Present Study	6.6061	6.5493	6.4582	6.1934	5.6769	5.3107	3.8066	3.2945	1.7069

According to literature, shells with negative Gaussian curvature will exhibit the lowest frequencies compared with their non-negative Gaussian counterparts. Thus, it is intended further investigations about the curvature effects of composite shells with positive and negative gaussian curvature. Thus, the proposed solution methodology with GDQ is compared with the obtained FEM results (ANSYS) in Table 6.3 for antisymmetric cross-ply [0°/90°] laminated negative Gaussian curvature panel, (R1= -R or R2= -R),and spherical panel (R1= R, R2= R) and plate for a/h=20 and 50.

Table 6.3 : Comparison of frequency parameter, λ of simply supported cross-ply [0°/90°] laminated plates and panels with positive and negative Gaussian curvature for different a/h ratios.

	$R_1/a=R_2/a=\infty$		$R_1/a=R_2/a=10$		$R_1/a=10$, $R_2/a=-10$		$R_1/a=-10$, $R_2/a=10$	
a/h	GDQ	FEA	GDQ	FEA	GDQ	FEA	GDQ	FEA
20	7543.4	7360.7	7832.5	7657.2	7553.8	7341.1	7500.2	7358.5
50	3076.4	3063.1	3758.7	3748.1	3074.2	3057.4	3065.1	3060.5

As can be seen from Table 6.3, mathematical model with the GDQ can effectively calculate for panels with saddle shapes. Remarkably, the results are very similar between GDQ and FEM for all types of panels, indicating the theory's capability to handle these geometries accurately.

The following mechanical properties [9] are assumed for convergence fundamental frequencies study of composite cross-ply laminate panel:

Mat Type 2: $E_1 = 175.78$ *GPa*, $E_1/E_2 = 25$, $G_{12} = G_{13} = 0.5E_2$, $G_{23} = 0.2E_2$, *and* $v_{12} = 0.25$.

Since the stability and accuracy are highly affected by the count of grid point, a convergence study is studied firstly. A rapid and monotonic convergence is observed for grid number m, n=9x9 for SS3 type boundary conditions as can be shown in Table 6.4.

Table 6.4: Convergence study of first three natural frequencies (Hz) of cross-ply composite laminated [0°/90°] panel according to different grid numbers (a/h=20) (SS3 Boundary Condition).

	Grid number	7x7	9x9	11x11	13x13	15x15	17x17	19x19	21x21	23x23
Mode	1	3.4506	3.4485	3.4485	3.4485	3.4485	3.4485	3.4485	3.4485	3.4485
	2	8.7684	8.8275	8.8216	8.8217	8.8217	8.8217	8.8217	8.8217	8.8217
	3	8.8159	8.8751	8.8692	8.8693	8.8693	8.8693	8.8693	8.8693	8.8693

Table 6.5 provide outcomes for antisymmetric [0°/90°] and symmetric cross-ply [0°/90°/0°] laminated negative Gaussian curvature panel, (R_1= -R or R_2= -R),and spherical panel (R_1= R, R_2= R) and plate with varying boundary conditions for different a/h ratios. Insights drawn from Table 6.5 reveal the following: Shells characterized by negative Gaussian curvature ,exhibit lower frequencies compared to shells with non-negative Gaussian curvature for cross-ply

composite laminated [0°/90°] for SS3, SS3-RS4, Mixed SS. However, panels with negative Gaussian curvature exhibit lower frequencies compared to shells with non-negative Gaussian curvature for cross-ply composite laminated [0°/90°/0°] for SS3, SS3-RS4, Mixed SS. Regardless of boundary conditions, the nondimensional fundamental frequencies are increased by increasing the side-to-thickness ratio a/h.

Table 6.5: Nondimensionalized frequency parameter of cross-ply laminated plates and panels with positive and negative Gaussian curvature for different a/h ratios.

		$R_1/a=R_2/a=\infty$		$R_1/a=R_2/a=10$		$R_1/a=10$, $R_2/a=-10$		$R_1/a=-10$, $R_2/a=10$	
B.C.	a/h	[0°/90°]	[0°/90°/0°]	[0°/90°]	[0°/90°/0°]	[0°/90°]	[0°/90°/0°]	[0°/90°]	[0°/90°/0°]
SS3	20	3.321	4.988	3.448	5.070	3.326	4.978	3.302	4.977
	50	3.386	5.276	4.137	5.779	3.383	5.265	3.374	5.267
	100	3.396	5.322	5.856	7.132	3.390	5.311	3.386	5.311
SS3-RS4	20	2,963	2,459	3.178	3,867	2,974	3,856	4,023	3,856
	50	3,265	2,942	4.735	6,257	4,3048	5,943	6,721	5,943
	100	3,332	3,075	8.300	9,127	7,027	8,231	8,716	8,231
Mixed SS	20	3,426	4,988	3,513	5,080	3,326	5,032	3,659	5,032
	50	3,503	5,276	4,255	5,879	3,545	5,618	4,387	5,618
	100	3,515	5,322	6,141	7,457	4,553	6,607	5,787	6,607

4. CONCLUSIONS

In this study, frequency analysis of moderately thick laminated cylindrical, spherical and negative Gaussian curvature panel for various combinations of boundary conditions was presented using GDQ method. The governing equilibrium equations for laminated composite panels were obtained using the virtual work principle. Comprehensive tabular and graphical results were displayed to show the effects of curvature, stacking lamination and boundary conditions on the frequency response of plate and panel structures. The validity of the proposed model was authenticated with a review of the available literature, and the convergence characteristics
were demonstrated. The numerical results could particularly be utilized during the early design stages of such laminated structures and as benchmark solutions for the future comparison of numerical results.

REFERENCES

J.N. Reddy, Mechanics of laminated composite plates and shells: theory and analysis, 2nd ed., CRC press, 2004.

H. Kurtaran, Geometrically nonlinear transient analysis of moderately thick laminated composite shallow shells with generalized differential quadrature method, Compos. Struct., 125, 605–614,2015.

İ. Algül and A.S. Oktem, Analytical and Numerical Solutions to Static Analysis of Moderately Thick Cross-Ply Plates and Shells, *Appl. Sci.*, *12*, 12547, 2022.

C. Shu and H. Du, A generalized approach for implementing general boundary conditions in the GDQ free vibration analysis of plates, Int. J. Solids Struct., 34, 837–846,1997.

Tornabene F., Fantuzzi N. and Bacciocchi M., The local GDQ method for the natural frequencies of doubly curved shells with variable thickness: a general formulation, Compos. B. Eng., 92, 265–289, 2016.

Liew K.M., Solving the vibration of thick symmetric laminates by Reissner/Mindlin plate theory and thep-Ritz method, J. Sound Vib., 198,343–360, 1996.

S. Srinivas, C.V.J. Rao and A.K. Rao, An exact analysis for vibration of simply supported homogeneous and laminated thick rectangular plates, J. Sound Vib., 12, 187–199, 1970.

J.N. Reddy and N. Phan, Stability and vibration of isotropic, orthotropic and laminated plates according to a higher-order shear deformation theory, J. Sound Vib., 98, 157–170, 1985.

A. S. Oktem and A. R. Chaudhuri, Levy type Fourier analysis of thick cross-ply doubly curved panels, Composite Structures, 80 (4), 475-488, 2007.

On Some Inequalities for Geometrically Convex Functions via Hadamard Fractional Integrals

Mehmet Zeki SARIKAYA[1]

[1]*Department of Mathematics, Faculty of Science and Arts,*
Düzce University, Düzce-TURKEY

sarikayamz@gmail.com
mzekisarikaya@duzce.edu.tr

ABSTRACT

In this study, we present a new generalization of the Hermite-Hadamard type inequalities for geometrically convex functions via Hadamard fractional integras. Also, we give some new inequalities for Hadamard fractional integrals by using two identities.

INTRODUCTION

Definition 1. *The function* $f : J \subseteq (0,\infty) \to (0,\infty)$ *is said to be* GG*-convex (geometrically convex) if the following inequality holds*

$$f(x^{\lambda} y^{(1-\lambda)}) \le [f(x)]^{\lambda} [f(y)]^{(1-\lambda)}$$

for all $x, y \in J$ *and* λ *in* $[0,1]$.

In 2013, Iscan [8] also proved the following result:

Theorem 1. *Suppose that* $f : I \subset \mathbb{R}^{+} \to \mathbb{R}^{+}$ *is geometrically convex and* $a, b \in I$ *with* $a < b$, *and* $f \in L[a,b]$. *Then*

$$f\left(\sqrt{ab}\right) \le \frac{1}{(\ln b - \ln a)} \int_a^b \sqrt{f(z) f\left(\frac{ab}{z}\right)} \frac{dz}{z}$$
$$\le \frac{1}{(\ln b - \ln a)} \int_a^b f(z) \frac{dz}{z} \le L(f(a), f(b)) \le \frac{f(a) + f(b)}{2} \tag{1.1}$$

where the logarithmic mean $L(u,v)$ *of two positive numbers* u, v *by*

$$L(u,v) := \begin{cases} \frac{u-v}{\ln u - \ln v}, & \text{if } u \ne v \\ u, & \text{if } u = v. \end{cases}$$

In addition to the convex function, many authors are working on geometric convexity, and with this definition, many new inequalities of the Hermite-Hadamard type are being studied, please see in the references [1], [8]-[14].

While fractional calculus has a rich historical background, recent developments in the field, particularly in the introduction of novel fractional derivative and integral operators by researchers, have revitalized interest in this area, particularly within applied sciences. This surge in interest has led to the introduction of numerous new fractional operators into the literature, driven by investigations into the properties of fractional derivative and associated integral operators, such as their singularity and locality, and modifications to their kernel structure.

Another type of fractional derivative that is mentioned in the literature is the Hadamard fractional derivative, which was introduced by Hadamard in 1892, see [4] and [5]. It distinguishes itself from the Riemann-Liouville and Caputo derivatives in that the integral kernel contains a logarithmic function with an arbitrary exponent. More information about the Hadamard fractional derivative and its properties can be found in references. Recent literature has also featured results related to fractional integral inequalities employing the Hadamard fractional integral. Therefore, the study of fractional differential equations need more developmental of inequalities of fractional type, for some of them, please see ([2], [3], [6], [7], [14]-[16]).

Definition 2. *Let* $f \in L_1[a,b]$. *The left and righet Hadamard fractional integrals* $H^{\alpha}_{a+}f$ *and* $H^{\alpha}_{b-}f$ *of order* $\alpha > 0$ *with* $a \geq 1$ *are defined by*

$$H^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \left(\ln\frac{x}{t}\right)^{\alpha-1} f(t)\frac{dt}{t}, \quad x > a$$

and

$$H^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b \left(\ln\frac{t}{x}\right)^{\alpha-1} f(t)\frac{dt}{t}, \quad x < b$$

respectively where $\Gamma(\alpha) = \int_0^\infty e^{-t}u^{\alpha-1}du$.

In this study, the following definitions will be made specifically:

$$H^{\alpha}_{a+}\left(\sqrt{f(b)f(a)}\right) = \frac{1}{\Gamma(\alpha)}\int_a^b \left(\ln\frac{z}{a}\right)^{\alpha-1}\sqrt{f(z)f\left(\frac{ab}{z}\right)}\frac{dz}{z},$$
$$H^{\alpha}_{b-}\left(\sqrt{f(b)f(a)}\right) = \frac{1}{\Gamma(\alpha)}\int_a^b \left(\ln\frac{b}{z}\right)^{\alpha-1}\sqrt{f(z)f\left(\frac{ab}{z}\right)}\frac{dz}{z}.$$

In this paper, we introduce a novel extension of the Hermite-Hadamard inequalities for geometrically convex functions via Hadamard fractional integrals. In order to exemplify its principal findings, a novel identity will be derived, and on the basis of said identity, some new integral inequalities will be presented. Additionally, we derive new inequalities that have strong connections with the right and left hand sides of the Hermite-Hadamard inequalities for Hadamard fractional integrals.

MAIN RESULTS

First, let's start our article by obtaining the Hermite-Hadamard inequality for the geometrically convex function.

Theorem 2. *Let* $f\ :\ [a,b]\subseteq(0,\infty)\to(0,\infty)$ *be a geometrically convex function on* $[a,b]$. *If* $f\in L[a,b]$, *then the following inequalities hold:*

$$f\left(\sqrt{ab}\right)\le\frac{\Gamma(\alpha+1)}{2\left(\ln\frac{b}{a}\right)^{\alpha}}\left[H_{a^+}^{\alpha}\left(\sqrt{f(b)f(a)}\right)+H_{b^-}^{\alpha}\left(\sqrt{f(b)f(a)}\right)\right]$$

$$\le\frac{\Gamma(\alpha+1)}{2\left(\ln\frac{b}{a}\right)^{\alpha}}\left[H_{a^+}^{\alpha}f(b)+H_{b^-}^{\alpha}f(a)\right]$$

$$\le\frac{\alpha}{2}\left[\frac{f(b)}{\left[\ln\frac{f(a)}{f(b)}\right]^{\alpha}}\int_1^{\frac{f(a)}{f(b)}}(\ln x)^{\alpha-1}dx+\frac{f(a)}{\left[\ln\frac{f(b)}{f(a)}\right]^{\alpha}}\int_1^{\frac{f(b)}{f(a)}}(\ln x)^{\alpha-1}dx\right]$$

$$\le\frac{f(a)+f(b)}{2}. \tag{2.1}$$

Proof. Since f is geometrically convex function on $[a,b]$ and using geometric-aritmatic inequality, we have

$$f\left(\sqrt{xy}\right)\le\sqrt{[f(x)][f(y)]}\le\frac{f(x)+f(y)}{2}.$$

For $t\in[0,1]$, $x=a^{1-t}b^{t}$, $y=a^{t}b^{1-t}\in[a,b]$, it follows that

$$f\left(\sqrt{ab}\right)\le\sqrt{\left[f\left(a^{1-t}b^{t}\right)\right]\left[f\left(a^{t}b^{1-t}\right)\right]}\le\frac{f\left(a^{1-t}b^{t}\right)+f\left(a^{t}b^{1-t}\right)}{2}.$$

By multiplying the result by $t^{\alpha-1}$ and integrates both sides of the inequality according to the parameter t on $[0,1]$, we get

$$2f\left(\sqrt{ab}\right)\int_0^1 t^{\alpha-1}dt\le 2\int_0^1 t^{\alpha-1}\sqrt{\left[f\left(a^{1-t}b^{t}\right)\right]\left[f\left(a^{t}b^{1-t}\right)\right]}dt$$

$$\le\int_0^1 t^{\alpha-1}f\left(a^{1-t}b^{t}\right)dt+\int_0^1 t^{\alpha-1}f\left(a^{t}b^{1-t}\right)dt.$$

Using the change of the variable, we obtain that

$$
\begin{aligned}
&\frac{2}{\alpha} f\left(\sqrt{ab}\right) \\
&\leq \frac{1}{\left(\ln \frac{b}{a}\right)^{\alpha}} \int_a^b \left(\ln \frac{z}{a}\right)^{\alpha-1} \sqrt{f(z) f\left(\frac{ab}{z}\right)} \frac{dz}{z} \\
&+ \frac{1}{\left(\ln \frac{b}{a}\right)^{\alpha}} \int_a^b \left(\ln \frac{b}{z}\right)^{\alpha-1} \sqrt{f(z) f\left(\frac{ab}{z}\right)} \frac{dz}{z} \\
&\leq \frac{1}{\left(\ln \frac{b}{a}\right)^{\alpha}} \int_a^b \left(\ln \frac{z}{a}\right)^{\alpha-1} f(z) \frac{dz}{z} + \frac{1}{\left(\ln \frac{b}{a}\right)^{\alpha}} \int_a^b \left(\ln \frac{b}{z}\right)^{\alpha-1} f(z) \frac{dz}{z} \\
&= \frac{\Gamma(\alpha)}{\left(\ln \frac{b}{a}\right)^{\alpha}} \left[H_{a^+}^{\alpha} f(b) + H_{b^-}^{\alpha} f(a) \right]
\end{aligned}
$$

where

$$
\int_a^b \left(\ln \frac{z}{a}\right)^{\alpha-1} \sqrt{f(z) f\left(\frac{ab}{z}\right)} \frac{dz}{z} = \int_a^b \left(\ln \frac{b}{z}\right)^{\alpha-1} \sqrt{f(z) f\left(\frac{ab}{z}\right)} \frac{dz}{z}.
$$

This is the first part of inequalities (2.1). On the other hand, since f is geometrically convex function on $[a,b]$, we have

$$
f\left(a^t b^{1-t}\right) \leq [f(a)]^t [f(b)]^{1-t} \leq t f(a) + (1-t) f(b),
$$

$$
f\left(a^{1-t} b^t\right) \leq [f(a)]^{1-t} [f(b)]^t \leq t f(a) + (1-t) f(b).
$$

By adding two these inequalites, we have

$$
\left[f\left(a^t b^{1-t}\right) + f\left(a^{1-t} b^t\right)\right] \leq f(b) \left[\frac{f(a)}{f(b)}\right]^t + f(a) \left[\frac{f(b)}{f(a)}\right]^t \leq f(a) + f(b)
$$

and multiplying the result by $t^{\alpha-1}$ and integrates both sides of the inequality according to the parameter t on $[0,1]$, it follows that

$$
\begin{aligned}
&\int_0^1 t^{\alpha-1} f\left(a^t b^{1-t}\right) dt + \int_0^1 t^{\alpha-1} f\left(a^{1-t} b^t\right) dt \\
&\leq f(b) \int_0^1 \left[\frac{f(a)}{f(b)}\right]^t t^{\alpha-1} dt + f(a) \int_0^1 \left[\frac{f(b)}{f(a)}\right]^t t^{\alpha-1} dt \\
&\leq \int_0^1 t^{\alpha-1} [f(a) + f(b)] dt.
\end{aligned}
$$

Using the change of the variable, we get

$$\frac{1}{(\ln\frac{b}{a})^{\alpha}}\int_a^b\left(\ln\frac{z}{a}\right)^{\alpha-1}f(z)\frac{dz}{z}+\frac{1}{(\ln\frac{b}{a})^{\alpha}}\int_a^b\left(\ln\frac{b}{z}\right)^{\alpha-1}f(z)\frac{dz}{z}$$

$$\leq\frac{f(b)}{\left[\ln\frac{f(a)}{f(b)}\right]^{\alpha}}\int_1^{\frac{f(a)}{f(b)}}(\ln x)^{\alpha-1}dx+\frac{f(a)}{\left[\ln\frac{f(b)}{f(a)}\right]^{\alpha}}\int_1^{\frac{f(b)}{f(a)}}(\ln x)^{\alpha-1}dx$$

$$\leq\frac{f(a)+f(b)}{\alpha}.$$

Thus, we obtain desired the second part of inequalities (2.1).

Remark 1. *In Theorem 2, if we choose* $\alpha=1$, *then the inequality (2.1) becomes the inequality (1.1).*

To prove our other main results, we require the following lemma:

Lemma 1. *Let* $f: I\subset\mathbb{R}^{+}\to\mathbb{R}+$ *be differentiable function on* I°, *the interior of the interval* I, *where* $a,b\in I$ *with* $a<b$, *and* $f'\in L[a,b]$. *Then the following identity holds,*

$$\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(\ln\frac{b}{a})^{\alpha}}\left[H_{a^+}^{\alpha}f(b)+H_{b^-}^{\alpha}f(a)\right]\tag{2.2}$$

$$=\frac{a}{2}\ln\frac{b}{a}\int_0^1 t^{\alpha}\left(\frac{b}{a}\right)^{t}f'\left(a^{1-t}b^{t}\right)dt-\frac{b}{2}\ln\frac{b}{a}\int_0^1 t^{\alpha}\left(\frac{a}{b}\right)^{t}f'\left(a^{t}b^{1-t}\right)dt.$$

Proof. By integration by parts, we have

$$\int_0^1 t^{\alpha}\left(\frac{b}{a}\right)^{t}f'\left(a^{1-t}b^{t}\right)dt=\frac{1}{a\ln\frac{b}{a}}f(b)-\frac{\alpha}{a(\ln\frac{b}{a})}\int_0^1 t^{\alpha-1}f\left(a^{1-t}b^{t}\right)dt$$

and

$$\int_0^1 t^{\alpha}\left(\frac{a}{b}\right)^{t}f'\left(a^{t}b^{1-t}\right)dt=-\frac{1}{b\ln\frac{b}{a}}f(a)+\frac{\alpha}{b(\ln\frac{b}{a})}\int_0^1 t^{\alpha-1}f\left(a^{t}b^{1-t}\right)dt.$$

By subtract by side to side these two integrals and using the change of the variable, we have

$$a\int_0^1 t^{\alpha}\left(\frac{b}{a}\right)^t f'\left(a^{1-t}b^t\right)dt-b\int_0^1 t^{\alpha}\left(\frac{a}{b}\right)^t f'\left(a^t b^{1-t}\right)dt$$

$$=\frac{1}{\ln\frac{b}{a}}[f(a)+f(b)]-\frac{\alpha}{\left(\ln\frac{b}{a}\right)}\left[\int_0^1 t^{\alpha-1}f\left(a^{1-t}b^t\right)dt+\int_0^1 t^{\alpha}\left(\frac{a}{b}\right)^t f'\left(a^t b^{1-t}\right)dt\right]$$

$$=\frac{1}{\ln\frac{b}{a}}[f(a)+f(b)]-\frac{\Gamma(\alpha+1)}{\left(\ln\frac{b}{a}\right)^{\alpha+1}}\left[\frac{1}{\Gamma(\alpha)}\int_a^b\left(\ln\frac{z}{a}\right)^{\alpha-1}f(z)\frac{dz}{z}+\frac{1}{\Gamma(\alpha)}\int_a^b\left(\ln\frac{b}{z}\right)^{\alpha-1}f(z)\frac{dz}{z}\right].$$

By multiplying this result by $\frac{1}{2}\ln\frac{b}{a}$, it is desired equality (2.2).

Remark 2. *In Lemma 1, we choose* $\alpha=1$ *, then the equality (2.2) becomes the following equality,*

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(z)\frac{dz}{z}=\frac{\ln\frac{b}{a}}{2}\left\{a\int_0^1 t\left(\frac{b}{a}\right)^t f'\left(a^{1-t}b^t\right)dt-b\int_0^1 t\left(\frac{a}{b}\right)^t f'\left(a^t b^{1-t}\right)dt\right\}$$

which is proved by İşcan in [8].

Theorem 3. *With the assumptions in Lemma 1. If* $|f'|$ *is geometrically convex on* $[a,b]$ *, then we have the following inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2\left(\ln\frac{b}{a}\right)^{\alpha}}\left[H_{a^+}^{\alpha}f(b)+H_{b^-}^{\alpha}f(a)\right]\right|$$

$$\leq\frac{1}{2}\ln\frac{b}{a}\left[a|f'(a)|G_{\alpha}\left(\frac{b|f'(b)|}{a|f'(a)|}\right)+b|f'(b)|G_{\alpha}\left(\frac{a|f'(a)|}{b|f'(b)|}\right)\right] \tag{2.3}$$

where G_{α} *is defined by*

$$G_{\alpha}(s)=\frac{1}{[\ln(s)]^{\alpha+1}}\int_1^s(\ln u)^{\alpha}du.$$

Proof. We take absolute value of (2.2) and by using the geometrically convexity of $|f'|$, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2\left(\ln\frac{b}{a}\right)^{\alpha}}\left[H_{a^+}^{\alpha}f(b)+H_{b^-}^{\alpha}f(a)\right]\right|$$

$$\leq \frac{a}{2}\ln\frac{b}{a}\int_0^1 t^{\alpha}\left(\frac{b}{a}\right)^t \left|f'\left(a^{1-t}b^t\right)\right|dt+\frac{b}{2}\ln\frac{b}{a}\int_0^1 t^{\alpha}\left(\frac{a}{b}\right)^t \left|f'\left(a^t b^{1-t}\right)\right|dt$$

$$\leq \frac{a}{2}\ln\frac{b}{a}\int_0^1 t^{\alpha}\left(\frac{b}{a}\right)^t |f'(a)|^{1-t}|f'(b)|^t dt+\frac{b}{2}\ln\frac{b}{a}\int_0^1 t^{\alpha}\left(\frac{a}{b}\right)^t |f'(a)|^t|f'(b)|^{1-t} dt$$

$$=\frac{a}{2}\ln\frac{b}{a}\frac{|f'(a)|}{\left[\ln\left(\frac{b|f'(b)|}{a|f'(a)|}\right)\right]^{\alpha+1}}\int_1^{\frac{b|f'(b)|}{a|f'(a)|}}(\ln u)^{\alpha}du+\frac{b}{2}\ln\frac{b}{a}\frac{|f'(b)|}{\left[\ln\left(\frac{a|f'(a)|}{b|f'(b)|}\right)\right]^{\alpha+1}}\int_1^{\frac{a|f'(a)|}{b|f'(b)|}}(\ln u)^{\alpha}du.$$

This proves the inequality (2.3).

Corollary 1. *Under the conditions of the Theorem 3 with* $\alpha=1$, *we ave the following inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{\ln b-\ln a}\int_a^b f(x)\frac{dx}{x}\right|\leq\frac{1}{2}\ln\frac{b}{a}L\big(b|f'(b)|,\ a|f'(a)|\big). \qquad (2.4)$$

Proof. In Theorem 3, if we choose $\alpha=1$, then the inequality (2.3) becomes the following inequality

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{\ln b-\ln a}\int_a^b f(x)\frac{dx}{x}\right|$$

$$\leq\frac{a}{2}\ln\frac{b}{a}\frac{|f'(a)|}{\left[\ln\left(\frac{b|f'(b)|}{a|f'(a)|}\right)\right]^2}\int_1^{\frac{b|f'(b)|}{a|f'(a)|}}\ln u\,du+\frac{b}{2}\ln\frac{b}{a}\frac{|f'(b)|}{\left[\ln\left(\frac{a|f'(a)|}{b|f'(b)|}\right)\right]^2}\int_1^{\frac{a|f'(a)|}{b|f'(b)|}}\ln u\,du$$

$$=\frac{1}{2}\ln\frac{b}{a}\frac{1}{\ln\left(\frac{b|f'(b)|}{a|f'(a)|}\right)}\left\{\left[b|f'(b)|-\frac{b|f'(b)|-a|f'(a)|}{\ln b|f'(b)|-\ln a|f'(a)|}\right]\right.$$

$$\left.+\left[\frac{b|f'(b)|-a|f'(a)|}{\ln b|f'(b)|-\ln a|f'(a)|}-a|f'(a)|\right]\right\}$$

$$=\frac{1}{2}\ln\frac{b}{a}L\big(b|f'(b)|,a|f'(a)|\big)$$

which is proved the inequality (2.4).

Lemma 2. *Let* $f : I \subset \mathbb{R}^{+} \to \mathbb{R}^{+}$ *be differentiable function on* I°*, the interior of the interval* I*, where* $a,b \in I$ *with* $a<b$*, and* $f' \in L[a,b]$*. Then the following identity holds,*

$$f\left(\sqrt{ab}\right)-\frac{\Gamma(\alpha+1)}{2\left(\ln\frac{b}{a}\right)^{\alpha}}\left[H_{a^{+}}^{\alpha} f(b)+H_{b^{-}}^{\alpha} f(a)\right] \tag{2.5}$$

$$=\frac{1}{2}\ln\frac{b}{a}\left\{a\int_{0}^{\frac{1}{2}} t^{\alpha}\left(\frac{b}{a}\right)^{t} f'\left(a^{1-t}b^{t}\right)dt-b\int_{0}^{\frac{1}{2}} t^{\alpha}\left(\frac{a}{b}\right)^{t} f'\left(a^{t}b^{1-t}\right)dt\right.$$

$$\left.+a\int_{\frac{1}{2}}^{1}\left(t^{\alpha}-1\right)\left(\frac{b}{a}\right)^{t} f'\left(a^{1-t}b^{t}\right)dt+b\int_{\frac{1}{2}}^{1}\left(1-t^{\alpha}\right)\left(\frac{b}{a}\right)^{t} f'\left(a^{t}b^{1-t}\right)dt\right\}.$$

Proof. By integration by parts, we have

$$a\int_{0}^{\frac{1}{2}} t^{\alpha}\left(\frac{b}{a}\right)^{t} f'\left(a^{1-t}b^{t}\right)dt=\frac{1}{2^{\alpha}}\frac{1}{\ln\frac{b}{a}} f\left(\sqrt{ab}\right)-\frac{\alpha}{\left(\ln\frac{b}{a}\right)}\int_{0}^{\frac{1}{2}} t^{\alpha-1} f\left(a^{1-t}b^{t}\right)dt,$$

$$-b\int_{0}^{\frac{1}{2}} t^{\alpha}\left(\frac{a}{b}\right)^{t} f'(a^{t}b^{1-t})dt = \frac{1}{2^{\alpha}}\frac{1}{\ln\frac{b}{a}} f\left(\sqrt{ab}\right)-\frac{\alpha}{\left(\ln\frac{b}{a}\right)}\int_{0}^{\frac{1}{2}} t^{\alpha-1} f(a^{t}b^{1-t})dt,$$

$$a\int_{\frac{1}{2}}^{1}\left(t^{\alpha}-1\right)\left(\frac{b}{a}\right)^{t} f'\left(a^{1-t}b^{t}\right)dt=\left(1-\frac{1}{2^{\alpha}}\right)\frac{1}{\ln\frac{b}{a}} f\left(\sqrt{ab}\right)-\frac{\alpha}{\left(\ln\frac{b}{a}\right)}\int_{\frac{1}{2}}^{1} t^{\alpha-1} f\left(a^{1-t}b^{t}\right)dt$$

and

$$b\int_{\frac{1}{2}}^{1}\left(1-t^{\alpha}\right)\left(\frac{b}{a}\right)^{t} f'\left(a^{t}b^{1-t}\right)dt=\left(1-\frac{1}{2^{\alpha}}\right)\frac{1}{\ln\frac{b}{a}} f\left(\sqrt{ab}\right)-\frac{\alpha}{\left(\ln\frac{b}{a}\right)}\int_{\frac{1}{2}}^{1} t^{\alpha-1} f\left(a^{t}b^{1-t}\right)dt.$$

By adding by side to side these four integrals and using the change of the variable, we have

$$a\int_0^{\frac{1}{2}} t^{\alpha}\left(\frac{b}{a}\right)^t f'\left(a^{1-t}b^t\right)dt - b\int_0^{\frac{1}{2}} t^{\alpha}\left(\frac{a}{b}\right)^t f'\left(a^t b^{1-t}\right)dt$$

$$+a\int_{\frac{1}{2}}^{1}\left(t^{\alpha}-1\right)\left(\frac{b}{a}\right)^t f'\left(a^{1-t}b^t\right)dt + b\int_{\frac{1}{2}}^{1}\left(1-t^{\alpha}\right)\left(\frac{b}{a}\right)^t f'\left(a^{1-t}b^t\right)dt$$

$$=\frac{2}{\ln\frac{b}{a}} f\left(\sqrt{ab}\right) - \frac{\Gamma(\alpha+1)}{\left(\ln\frac{b}{a}\right)^{\alpha+1}}\left[\frac{1}{\Gamma(\alpha)}\int_a^b\left(\ln\frac{z}{a}\right)^{\alpha-1} f(z)\frac{dz}{z} + \frac{1}{\Gamma(\alpha)}\int_a^b\left(\ln\frac{b}{z}\right)^{\alpha-1} f(z)\frac{dz}{z}\right].$$

By multiplying this result by $\frac{1}{2}\ln\frac{b}{a}$, it is desired equality (2.5).

Theorem 4. *With the assumptations in Lemma 2. If $|f'|$ is geometrically convex on $[a,b]$, then we have the following inequality*

$$\left|f\left(\sqrt{ab}\right) - \frac{\Gamma(\alpha+1)}{2\left(\ln\frac{b}{a}\right)^{\alpha}}\left[H_{a^+}^{\alpha} f(b) + H_{b^-}^{\alpha} f(a)\right]\right|$$

$$\leq \frac{1}{2}\ln\frac{b}{a}\left\{a|f'(a)|\left[T_{\alpha}\left(\frac{b|f'(b)|}{a|f'(a)|}\right) - Y_{\alpha}\left(\frac{b|f'(b)|}{a|f'(a)|}\right)\right]\right.$$

$$\left.+b|f'(b)|\left[T_{\alpha}\left(\frac{a|f'(a)|}{b|f'(b)|}\right) - Y_{\alpha}\left(\frac{a|f'(a)|}{b|f'(b)|}\right)\right]\right\}$$

$$+\frac{1}{2}\ln\frac{b}{a}\left[\frac{b|f'(b)| - a|f'(a)|}{\ln b|f'(b)| - \ln a|f'(a)|}\right] \tag{2.6}$$

where T_{α} *and* Y_{α} *are defined by*

$$T_{\alpha}(s) = \frac{1}{[\ln(s)]^{\alpha+1}}\int_1^{\sqrt{s}} (\ln u)^{\alpha}\, du$$

and

$$Y_{\alpha}(s) = \frac{1}{[\ln(s)]^{\alpha+1}}\int_{\sqrt{s}}^{s} (\ln u)^{\alpha}\, du.$$

Proof. We take absolute value of (2.5) and by using the geometrically convexity of $|f'|$, we have

$$\left|f\left(\sqrt{ab}\right)-\frac{\Gamma(\alpha+1)}{2\left(\ln\frac{b}{a}\right)^{\alpha}}\left[H_{a^+}^{\alpha}f(b)+H_{b^-}^{\alpha}f(a)\right]\right|$$

$$\leq\frac{1}{2}\ln\frac{b}{a}\left\{a\int_0^{\frac{1}{2}}t^{\alpha}\left(\frac{b}{a}\right)^{t}|f'(a)|^{1-t}|f'(b)|^{t}dt+b\int_0^{\frac{1}{2}}t^{\alpha}\left(\frac{a}{b}\right)^{t}|f'(a)|^{t}|f'(b)|^{1-t}dt\right.$$

$$\left.+a\int_{\frac{1}{2}}^{1}(1-t^{\alpha})\left(\frac{b}{a}\right)^{t}|f'(a)|^{1-t}|f'(b)|^{t}dt+b\int_{\frac{1}{2}}^{1}(1-t^{\alpha})\left(\frac{b}{a}\right)^{t}|f'(a)|^{t}|f'(b)|^{1-t}dt\right\}$$

$$=\frac{a}{2}|f'(a)|\ln\frac{b}{a}\left\{\int_0^{\frac{1}{2}}t^{\alpha}\left(\frac{b|f'(b)|}{a|f'(a)|}\right)^{t}dt+\int_{\frac{1}{2}}^{1}(1-t^{\alpha})\left(\frac{b|f'(b)|}{a|f'(a)|}\right)^{t}dt\right\}$$

$$+\frac{b}{2}|f'(b)|\ln\frac{b}{a}\left\{\int_0^{\frac{1}{2}}t^{\alpha}\left(\frac{a|f'(a)|}{b|f'(b)|}\right)^{t}dt+\int_{\frac{1}{2}}^{1}(1-t^{\alpha})\left(\frac{a|f'(a)|}{b|f'(b)|}\right)^{t}dt\right\}$$

$$=\frac{a}{2}\ln\frac{b}{a}\frac{|f'(a)|}{\left[\ln\frac{b|f'(b)|}{a|f'(a)|}\right]^{\alpha+1}}\left\{\int_1^{\sqrt{\frac{b|f'(b)|}{a|f'(a)|}}}(\ln u)^{\alpha}du+\int_{\sqrt{\frac{b|f'(b)|}{a|f'(a)|}}}^{\frac{b|f'(b)|}{a|f'(a)|}}\left[\left(\ln\frac{b|f'(b)|}{a|f'(a)|}\right)^{\alpha}-(\ln u)^{\alpha}\right]du\right\}$$

$$+\frac{b}{2}\frac{|f'(b)|}{\left[\ln\frac{a|f'(a)|}{b|f'(b)|}\right]^{\alpha+1}}\ln\frac{b}{a}\left\{\int_1^{\sqrt{\frac{a|f'(a)|}{b|f'(b)|}}}(\ln u)^{\alpha}du+\int_{\sqrt{\frac{a|f'(a)|}{b|f'(b)|}}}^{\frac{a|f'(a)|}{b|f'(b)|}}\left[\left(\ln\frac{a|f'(a)|}{b|f'(b)|}\right)^{\alpha}-(\ln u)^{\alpha}\right]du\right\}$$

This proves the inequality (2.6).

Corollary 2. *Under the conditions of the Theorem 4 with* $\alpha=1$, *we have the following inequality*

$$\left|f\left(\sqrt{ab}\right)-\frac{1}{\ln b-\ln a}\int_a^b f(x)\frac{dx}{x}\right|\leq\frac{1}{4}\ln\frac{b}{a}L^2\left(\sqrt{b|f'(b)|},\sqrt{a|f'(a)|}\right) \qquad (2.7)$$

Proof. In Theorem 4, if we choose $\alpha=1$, then the inequality (2.6) becomes the following inequality

$$\left| f\left(\sqrt{ab}\right) - \frac{1}{\ln b - \ln a}\int_a^b f(x)\frac{dx}{x} \right|$$

$$\leq \frac{1}{2}\ln\frac{b}{a}\left\{ a|f'(a)| \left[\frac{1}{\left[\ln\frac{b|f'(b)|}{a|f'(a)|}\right]^2}\int_1^{\sqrt{\frac{b|f'(b)|}{a|f'(a)|}}}(\ln u)du - \frac{1}{\left[\ln\frac{b|f'(b)|}{a|f'(a)|}\right]^2}\int_{\sqrt{\frac{b|f'(b)|}{a|f'(a)|}}}^{\frac{b|f'(b)|}{a|f'(a)|}}(\ln u)du \right]\right.$$

$$\left. + b|f'(b)| \left[\frac{1}{\left[\ln\frac{a|f'(a)|}{b|f'(b)|}\right]^2}\int_1^{\sqrt{\frac{a|f'(a)|}{b|f'(b)|}}}(\ln u)du - \frac{1}{\left[\ln\frac{a|f'(a)|}{b|f'(b)|}\right]^2}\int_{\sqrt{\frac{a|f'(a)|}{b|f'(b)|}}}^{\frac{a|f'(a)|}{b|f'(b)|}}(\ln u)du \right]\right\}$$

$$+ \frac{1}{2}\ln\frac{b}{a}\left[\frac{b|f'(b)| - a|f'(a)|}{\ln b|f'(b)| - \ln a|f'(a)|}\right]$$

$$= \frac{1}{2}\ln\frac{b}{a}\left\{ \frac{a|f'(a)|}{\left[\ln\frac{b|f'(b)|}{a|f'(a)|}\right]^2}\left[2\sqrt{\frac{b|f'(b)|}{a|f'(a)|}}\left(\ln\sqrt{\frac{b|f'(b)|}{a|f'(a)|}} - 1\right) + 1 - \frac{b|f'(b)|}{a|f'(a)|}\left(\ln\frac{b|f'(b)|}{a|f'(a)|} - 1\right)\right]\right.$$

$$\left. + \frac{b|f'(b)|}{\left[\ln\frac{a|f'(a)|}{b|f'(b)|}\right]^2}\left[2\sqrt{\frac{a|f'(a)|}{b|f'(b)|}}\left(\ln\sqrt{\frac{a|f'(a)|}{b|f'(b)|}} - 1\right) + 1 - \frac{a|f'(a)|}{b|f'(b)|}\left(\ln\frac{a|f'(a)|}{b|f'(b)|} - 1\right)\right]\right\}$$

$$+ \frac{1}{2}\ln\frac{b}{a}\left[\frac{b|f'(b)| - a|f'(a)|}{\ln b|f'(b)| - \ln a|f'(a)|}\right]$$

$$= = \ln\frac{b}{a}\left[\frac{\sqrt{b|f'(b)|} - \sqrt{a|f'(a)|}}{\ln b|f'(b)| - \ln a|f'(a)|}\right]^2$$

which is proved the inequality (2.7).

REFERENCES

1. G. D. Anderson, M. K. Vamanamurthy and M. Vuorinen, *Generalized convexity and inequalities*, J. Math. Anal. Appl., 335, (2007), 1294-1308.

2. H. Budak, E. Pehlivan and P. Kosem, *On new extensions of Hermite-Hadamard inequalities for generalized fractional integrals*, Sahand Communications in Mathematical Analysis, 18(1), 73-88, (2021).

3. H. Budak, C. C. Bilişik and M. Z. Sarikaya, *On some new extensions of inequalities of Hermite-Hadamard type for generalized fractional integrals*, Sahand Communications in Mathematical Analysis, 19(2), 65-79, (2022).

4. S. G. Samko, A. A Kilbas, O. I. Marichev, Fractional Integrals and Derivatives Theory and Application, Gordan and Breach Science, New York, 1993.

5. A.A. Kilbas, H.M. Srivastava and J.J.Trujillo, Theory and Applications of Fractional Differential Equations, vol.204 of North-Holland Mathematics Studies, Elsevier, Amsterdam, The Netherlands, 2006.

6. M.Z. Sarikaya, F. Ertugral, *On the generalized Hermite-Hadamard inequalities*, Annals of the University of Craiova-Mathematics and Computer Science Series 47 (2020), no. 1, 193-213.

7. M.Z. Sarikaya, H. Budak, *Generalized Hermite-Hadamard type integral inequalities for fractional integrals*, Filomat 30 (2016), no. 5, 1315--1326.

8. I. Iscan, *Some new Hermite-Hadamard type inequalities for geometrically convex functions*, Mathematics and Statistics, 1(2), (2013),86-91.

9. I. Iscan, *New general integral inequalities for quasi-geometrically convex functions via fractional integrals*, J. Inequal. Appl., 2013(491), (2013), 15 pages.

10. B.-Y. Xi, R.-F. Bai, F. Qi, *Hermite--Hadamard type inequalities for the* m*-and* (α, m)*-geometrically convex functions*, Aequat. Math. Vol. 84 (2012), 261--269.

11. T.-Y. Zhang, A.-P. Ji and F. Qi, *On integral inequalities of Hermite-Hadamard type for* s*-geometrically convex functions*, Abstract and Applied Analysis, Vol. 2012 (2012).

12. S.S. Dragomir, *Some integral inequalities of Hermite-Hadamard type for* GG*-convex functions*, RGMIA Res Rep Coll. 2017;59(82):1-2.

13. S.S. Dragomir, *Inequalities of Hermite-Hadamard type for* GG*-convex functions*, Analele Universitatii de Vest, Timisoara Seria Matematica-Informatica LVII,2,(2019),34-52.

14. M. Kunt and İ. İşcan, *On new inequalities of Hermite-Hadamard-Fejer type for* GA*-*s *convex functions via fractional integrals*, Konuralp Journal of Mathematics 4.1 (2016): 130-139.

15. J. Wang, X. Li, C. Zhu, *Refinements of Hermite-Hadamard type inequalities involving fractional integrals*, Bull. Belg. Math. Soc. Simon Stevin 20 (2013), 655--666.

16. J. Wang, C. Zhu and Y. Zhou, *New generalized Hermite-Hadamard type inequalities and applications to special means*, Journal of Inequalities and Applications, 2013(1), 1-15, (2013).

Stochastic Dynamics in Epidemic Modeling: Long-Term Analysis of an SIR Model Featuring Incidence Capping

Soulaimane AZNAGUE [1], Adel SETTATI [1], Aadil LAHROUZ [1] and Mourad EL IDRISSI [1]

[1] Laboratory of Mathematics and Applications, FST, Abdelmalek Essaadi University, Tetouan-MOROCCO

soulaimane.aznague@etu.uae.ac.ma
settati_adel@yahoo.fr
lahrouzadil@gmail.com
mourad.elidrissi@etu.uae.ac.ma

ABSTRACT

Our research investigates a stochastic SIR epidemic model characterized by a saturation effect in the incidence rate. The core of our analysis is the threshold dynamics, revealing that a negative $\lambda < 0$ enhances the probability of disease eradication. In contrast, a positive $\lambda > 0$ results in the model exhibiting a singular positive stationary distribution.

1. INTRODUCTION

In the early 1900s, epidemiology experienced a pivotal shift due to the influential work of renowned scientists like Anderson Gray McKendrick and Janet Leigh. They were instrumental in introducing mathematical modeling to epidemiology, an essential technique in the discipline. Mathematical models have significantly influenced how outbreaks and epidemics are managed, becoming a key factor in shaping data-driven public health strategies. The development of epidemiology, notably its transition into a formal scientific discipline, is marked by significant contributions from several key figures. Among them, Quinto Tiberio Angelerio is noteworthy for his adept handling of the plague in Alghero, Sardinia, in 1582. However, the birth of modern epidemiology is primarily attributed to the 19th century. John Snow, often hailed as the "father of modern epidemiology," played a crucial role in his detailed analysis of a cholera outbreak in London, tracing its source to the contaminated water of the Broad Street pump. This seminal investigation marks a turning point in epidemiology, establishing the foundations of the rigorous, data-driven discipline we know today. Mathematical models have become instrumental in revealing the complex dynamics of disease development and spread. A significant advancement in this area was the development of the SIR (Susceptible-Infected-Removed) epidemic model by Kermack and McKendrick in 1927 [1]. The incidence rate, a critical metric, quantifies new infections over a defined period. Various incidence rates have been explored to reflect the nuances of transmission dynamics accurately. These include the bilinear incidence rate (see, e.g., [2, 3, 4, 5, 6, 7] for more information and the references cited therein), saturation infection rate [8], and several nonlinear incidence rates (refer to [9, 10, 11] for further insights), each contributing to a more nuanced understanding of epidemiological patterns. Infectious diseases are sensitive to environmental variations, including temperature shifts, humidity, and broader climatic conditions. As a result, using stochastic differential equations has gained prominence in modeling disease transmission. These equations offer a more authentic portrayal of how stochastic elements influence the dynamics of infectious diseases. Our study focuses on a stochastic SIR epidemic model that incorporates a saturated incidence rate, offering insights into the complex interplay of disease transmission under stochastic influences.

$$\begin{cases} dS &= \left[\Lambda - \mu S - \frac{\beta SI}{1+\alpha S}\right] dt + \sigma_1 S dB_1, \\ dI &= \left[-(\mu+\gamma) I + \frac{\beta SI}{1+\alpha S}\right] dt + \sigma_2 I dB_2, \\ dR &= [\gamma I - \mu R] dt + \sigma_3 R dB_3. \end{cases} \tag{1.1}$$

At a given moment, the variables $S(t)$, $I(t)$, and $R(t)$ represent the number of susceptible, infected, and recovered individuals, respectively, in the population. Within this model, the positive parameters have specific interpretations: γ is the rate at which infected individuals recover, $\beta S/(1+\alpha S)$ describes the incidence rate with β as the rate of disease transmission, and α as the half-saturation constant. Additionally, μ signifies the natural mortality rate of the population, while Λ indicates the rate at which new individuals are added to the people. In our model, there are independent Brownian motions, represented by $B_i(t)$ where $i = 1,2,3$, each accompanied by a corresponding intensity of white noise, denoted as σ_i, and all these intensities are positive. Given that the final equation in system (1.1) operates independently of the first two, our analysis is concentrated exclusively on the subsequent equations:

$$\begin{cases} dS &= \left[\Lambda - \mu S - \frac{\beta SI}{1+\alpha S}\right] dt + \sigma_1 S dB_1, \\ dI &= \left[-(\mu+\gamma) I + \frac{\beta SI}{1+\alpha S}\right] dt + \sigma_2 I dB_2. \end{cases} \tag{1.2}$$

2. PRELIMINARIES

Throughout this paper, we consider the following notations: $\mathbb{R}_+^2 = \{(s,i): s \geq 0, i \geq 0\}$, $\mathbb{R}_+^{2,o} = \{(s,i): s > 0, i > 0\}$. We establish $(\Omega, \mathcal{T}, \{\mathcal{T}_t\}_{t\geq 0}, \mathbb{P})$ as a complete probability space, accompanied by a filtration $\{\mathcal{T}_t\}_{t\geq 0}$ that adheres to standard conditions. Our research focuses on the analysis of a d-dimensional Itô process governed by the following stochastic differential equation (SDE):

$$dX = f(X)dt + g(X)dB, \quad for\ each \quad t \geq t_0. \tag{2.1}$$

Here, $B(t)$ represents a $d-$ dimensional white noise, and the starting value $X(0) \in \mathbb{R}^d$. According to Itô formula, the stochastic equation verified by $V(X(t),t)$ where V is a function that is continuously twice differentiable and defined on $\mathbb{R}^d \times \mathbb{R}^+$, is given by:

$$dV(X) = \mathcal{L}V(X)dt + \nabla V(X) g(X) dB(t),$$

where $\nabla V = \left(\frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_d}\right)$ is gradient of V and $\mathcal{L}$ is the differential operator with system (2.1), defined by

$$\mathcal{L} = \frac{\partial}{\partial t} + \sum_{i=1}^{d} f_i(X)\frac{\partial}{\partial X_i} + \frac{1}{2}\sum_{i,j=1}^{d} [g^T(X)\cdot g(X)]_{ij}\frac{\partial^2}{\partial X_i \partial X_j}.$$

Theorem 2.1. *For each $X(0) \in \mathbb{R}_+^2$, there exists a unique positive solution X of SDE (1.2) for every $t \geq 0$ such that*

$$\mathbb{P}_{s,i}\{X(t) \in \mathbb{R}_+^2\} = 1.$$

Proof. The proof of positivity of solutions follows a methodology similar to that in [11], which we omit for brevity.
Next, we establish the threshold that governs the dynamics of the model (1.2). To proceed, we consider the following equation obtained by setting $I(t) = 0$ in the first equation of model (1.2) and denote the solution of this equation as $\hat{S}(t)$. Thus

$$d\hat{S}(t) = \left[\Lambda - \mu\hat{S}(t)\right]dt + \sigma_1\hat{S}(t)dB_1, \tag{2.2}$$

with initial value $S(0)$. It follows from the comparison theorem [12] that for all $t \leq 0$

$$S(t) \leq \hat{S}(t), \quad \text{a.s..}$$

As shown in [13], the process $\hat{S}(t)$ exhibits a unique stationary distribution $\boldsymbol{\pi}$ where the expression of its density is given, using Fokker-Planck equation by

$$f^*(x) = \frac{(2\Lambda\sigma_1^{-2})^{2\mu\sigma_1^{-2}+1}}{\Gamma\left(1+\frac{2\mu}{\sigma_1^2}\right)} x^{-(2+2\mu\sigma_1^{-2})}\exp\left(\frac{-2\Lambda}{\sigma_1^2 x}\right), \quad x \in (0,\infty),$$

where $\Gamma(\cdot)$ is a Gamma function. Therefore, one has

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \hat{S}(x)\,\mathrm{d}x = \int_0^\infty sf^*(s)\,\mathrm{d}s = \frac{\Lambda}{\mu} \quad a.s..$$

In the following, we shall show that the long-time behavior of $I(t)$ is determined by

$$\lambda = \int_0^\infty \frac{\beta s}{1+\alpha s}\boldsymbol{\pi}\,(\mathrm{d}s) - \mu - \gamma - \frac{\sigma_2^2}{2}.$$

3. MAIN RESULTS

The focal outcome of this section is the identification of nearly necessary and sufficient conditions determining the persistence and extinction of the disease.

Theorem 3.1. *Consider X as the solution to SDE (1.2) with an initial value X_0. Assume that $\lambda < 0$, then*

$$\limsup_{t\to\infty}\frac{1}{t}\ln I(t) \leq \lambda < 0, \quad a.s.. \tag{3.1}$$

If $\lambda > 0$, then, the model (1.2) possesses a unique stationary distribution $\upsilon^*(.)$ and is ergodic. Moreover,
(i) for any υ^*-integrable $f: \mathbb{R}_+^{2,o} \to \mathbb{R}$, we have

$$\mathbb{P}\left\{\lim_{t\to\infty}\frac{1}{t}\int_0^t f(X(\tau))\mathrm{d}\tau = \int_{\mathbf{R}_+^{2,o}} f(x)\upsilon^*(\mathrm{d}x)\right\} = 1.$$

(ii) For each $(s,i) \in \mathbb{R}_+^{2,o}$, $\lim_{t\to\infty} \parallel P(t,(s,i),\cdot) - \upsilon^*(\cdot) \parallel = 0$, where $P(t,(s,i),\cdot)$ is the transition probability.

Proof. **The case** $\lambda < 0$. Using the Itô formula to function $\ln I(t)$, we obtain

$$\begin{aligned} d(\ln I) &= \left[\frac{\beta S}{1+\alpha S} - \left(\gamma + \mu + \frac{\sigma_2^2}{2}\right)\right] dt + \sigma_2 dB_2, \\ &\le \left[\frac{\beta \hat{S}}{1+\alpha \hat{S}} - \left(\gamma + \mu + \frac{\sigma_2^2}{2}\right)\right] dt + \sigma_2 dB_2. \end{aligned}$$

By applying the strong law of large numbers and from the ergodicity of $\hat{S}(t)$, we obtain

$$\begin{aligned} \limsup_{t\to\infty} \frac{1}{t} \ln I(t) &\le \lim_{t\to\infty} \frac{1}{t} \int_0^t \frac{\beta \hat{S}}{1+\alpha \hat{S}}\, \mathrm{d}x - \mu - \gamma - \frac{\sigma_2^2}{2}, \\ &\le \int_0^\infty \frac{\beta s}{1+\alpha s} \boldsymbol{\pi}(\mathrm{d}s) - \mu - \gamma - \frac{\sigma_2^2}{2}, \\ &:= \lambda \quad \text{a.s..} \end{aligned}$$

This implies that

$$\lim_{t\to\infty} I(t) = 0 \quad \text{a.s..}$$

The proof is completed.

Proof. **The case** $\lambda > 0$. In view of system (1.2), we define

$$V_1 = -\ln I + \frac{\beta}{\mu}(\hat{S} - S) - \frac{\beta}{\mu} I.$$

Applying the Itô formula, we get

$$\begin{aligned} \mathcal{L}V_1 &= \left(\mu + \gamma + \frac{\sigma_2^2}{2}\right) - \frac{\beta S}{1+\alpha S} - \beta(\hat{S} - S) + \frac{\beta(\mu+\gamma)}{\mu} I, \\ &\le -\lambda + \int_0^\infty \frac{\beta s}{1+\alpha s} \boldsymbol{\pi}\,(\mathrm{d}s) - \frac{\beta \hat{S}}{1+\alpha \hat{S}} + \frac{\beta \hat{S}}{1+\alpha \hat{S}} - \frac{\beta S}{1+\alpha S} - \beta(\hat{S} - S) + \frac{\beta(\mu+\gamma)}{\mu} I, \\ &\le -\lambda + \int_0^\infty \frac{\beta s}{1+\alpha s} \boldsymbol{\pi}\,(\mathrm{d}s) - \frac{\beta \hat{S}}{1+\alpha \hat{S}} + \frac{\beta(\mu+\gamma)}{\mu} I. \end{aligned} \tag{3.2}$$

Define

$$V_2(S) = -\ln S, \quad V_3(X) = \frac{1}{m+1}(S+I)^{m+1},$$

where $m > 0$ satisfying

$$\theta := \mu - \frac{m}{2}(\sigma_1^2 \vee \sigma_2^2) > 0.$$

Applying the Itô formula to V_2 and V_3, respectively, we get

$$\mathcal{L}V_2 \le -\frac{\Lambda}{S} + \beta I + \mu + \frac{\sigma_1^2}{2}, \tag{3.3}$$

and

$$\begin{aligned} \mathcal{L}V_3 &= (S+I)^m[\Lambda - \mu S - (\mu+\gamma)I] + \frac{m}{2}(S+I)^{m-1}(\sigma_1^2 S^2 + \sigma_2^2 I^2), \\ &\le (S+I)^m \Lambda - \mu (S+I)^{m+1} + \frac{m}{2}(S+I)^{m+1}(\sigma_1^2 \vee \sigma_2^2), \\ &\le B - \frac{\theta}{2}(S+I)^{m+1}, \end{aligned} \tag{3.4}$$

where

$$B = \sup_{X\in\mathbb{R}_+^{2,o}} \left\{(S+I)^m\Lambda - \frac{\theta}{2}(S+I)^{m+1}\right\}.$$

We define

$$\tilde{V}(S,I) = CV_1(S,I) + V_2(S) + V_3(X),$$

where $C > 0$ is adequately large to fulfill the following condition:

$$-C\lambda + F \leq -2, \tag{3.5}$$

where

$$F = \sup_{S\in(0,\infty)} \left\{\beta I - \frac{\Lambda}{S} + B + \mu + \frac{\sigma_1^2}{2} - \frac{\theta}{2}S^{m+1} - \frac{\theta}{2}I^{m+1}\right\}.$$

Furthermore, it is essential to remark that $\tilde{V}(S,I)$ has a minimum value point (S^*,I^*). Then, we consider a non-negative C^2-function $W(S,I)$ as follows:

$$W(S,I) = \tilde{V}(S,I) - \tilde{V}(S^*,I^*).$$

In view of (3.2), (3.3) and (3.4), we obtain

$$\mathcal{L}W \leq -C\lambda + C\left(\int_0^\infty \frac{\beta s}{1+\alpha s}\boldsymbol{\pi}\,(\mathrm{d}s) - \frac{\beta\hat{S}}{1+\alpha\hat{S}}\right) + \frac{C\beta(\mu+\gamma)}{\mu}I - \frac{\Lambda}{S} + \beta I + \mu + \frac{\sigma_1^2}{2} + B - \frac{\theta}{2}S^{m+1} - \frac{\theta}{2}I^{m+1},$$

$$= G(S,I) + C\left(\int_0^\infty \frac{\beta s}{1+\alpha s}\boldsymbol{\pi}\,(\mathrm{d}s) - \frac{\beta\hat{S}}{1+\alpha\hat{S}}\right),$$

where

$$G(S,I) = -C\lambda + \frac{C\beta(\mu+\gamma)}{\mu}I - \frac{\Lambda}{S} + \beta I + \mu + \frac{\sigma_1^2}{2} + B - \frac{\theta}{2}S^{m+1} - \frac{\theta}{2}I^{m+1}.$$

Now, we can formulate a bounded open set $\mathcal{U}_\varepsilon$ as follows:

$$\mathcal{U}_\varepsilon = \left\{(S,I) \in \mathbb{R}_+^{2,o} /\quad \varepsilon < S < \frac{1}{\varepsilon},\quad \varepsilon < I < \frac{1}{\varepsilon}\right\},$$

where $0 < \varepsilon < 1$ is a sufficiently small. Furthermore, by using the explicit formulation of $G(S,I)$, one has

Case 1. Letting $S \to 0^+$ or $S \to \infty$ or $I \to \infty$, we get

$$G(S,I) = -C\lambda + \frac{C\beta(\mu+\gamma)}{\mu}I - \frac{\Lambda}{S} + \beta I + \mu + \frac{\sigma_1^2}{2} + B - \frac{\theta}{2}S^{m+1} - \frac{\theta}{2}I^{m+1},$$
$$\leq -\infty.$$

Case 2. Assume that $I \to 0^+$, we obtain

$$G(S,I) = -C\lambda - \frac{\Lambda}{S} + \beta I + \frac{C\beta(\mu+\gamma)}{\mu}I + \mu + \frac{\sigma_1^2}{2} + B - \frac{\theta}{2}S^{m+1} - \frac{\theta}{2}I^{m+1},$$
$$\leq -C\lambda + \frac{C\beta(\mu+\gamma)}{\mu}I + F \to -C\lambda + F,$$
$$\leq -2,$$

which follows from (3.5).

As a result, for a sufficiently small $\varepsilon > 0$, we obtain

$$G(X) \leq -1, \quad for\ all \quad (S,I) \in \mathbb{R}_+^{2,o}\backslash\mathcal{U}_\varepsilon. \tag{3.6}$$

In addition, there is also $T > 0$ such that

$$G(X) \leq T, \quad for\ each \quad (S,I) \in \mathbb{R}_+^{2,o}. \tag{3.7}$$

Hence, by the ergodicity of $\hat{S}$, combining (3.6) and (3.7), we obtain

$$0 \leq \liminf_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{E}(G(X(\tau)))\mathrm{d}\tau,$$

$$= \liminf_{t\to\infty}\frac{1}{t}\int_0^t \left[\mathbb{E}\left(G(X(\tau))\mathbb{I}_{\{X(\tau)\in\mathcal{U}_\varepsilon{}^c\}}\right) + \mathbb{E}\left(G(X(\tau))\mathbb{I}_{\{X(\tau)\in\mathcal{U}_\varepsilon\}}\right)\right]\mathrm{d}\tau,$$
$$\leq \liminf_{t\to\infty}\frac{1}{t}\int_0^t [TP(X(\tau)\in\mathcal{U}_\varepsilon) - P(X(\tau)\in\mathcal{U}_\varepsilon{}^c)]\mathrm{d}\tau,$$
$$= -1 + (T+1)\liminf_{t\to\infty}\frac{1}{t}\int_0^t P(X(\tau)\in\mathcal{U}_\varepsilon)\mathrm{d}\tau,$$

which implies

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t P(\tau,(s,i),\mathcal{U}_\varepsilon)\,\mathrm{d}\tau \geq \frac{1}{T+1}, \tag{3.8}$$

for all $X_0 = (s,i) \in \mathbb{R}_+^{2,o}$. Given that $\mathbb{R}_+^{2,o}$ is an invariant set according to (1.2), we can examine $X(t)$ within the state space $\mathbb{R}_+^{2,o}$. Additionally, leveraging the invariance of $\mathbb{R}_+^{2,o}$ and the inequality (3.8), it follows that there exists an invariant probability measure υ^* on $\mathbb{R}_+^{2,o}$, as established in [14]. Moreover, the independence of $B_i(t), i = 1,2$, as indicated in [15, 16], implies that $\mathbb{R}_+^{2,o}$ serves as the support of υ^*. In consideration of the results presented in references [17, 18, 19], it can be demonstrated that assertions (i) and (ii) are substantiated. This completes the proof.

4. CONCLUSION

In epidemiological research, exploring stochastic dynamics offers a nuanced perspective on disease spread and control. Our study, "Stochastic Dynamics in Epidemic Modeling: Long-Term Analysis of a SIR Model Featuring Incidence Capping," delves into the intricate interplay between random environmental factors and disease transmission. By integrating stochastic elements into the classical SIR (Susceptible-Infected-Recovered) model, this research sheds light on the long-term behavior of infectious diseases under varying conditions. A key feature of our model is the incorporation of incidence capping, which provides a more realistic representation of how external limitations and behavioral changes in a population may influence disease transmission. This approach allows for a deeper understanding of the unpredictable nature of epidemics and aids in developing more effective, data-driven public health strategies.

REFERENCES

[1] W. O. Kermack, A. G. McKendrick, A contribution to the mathematical theory of epidemics, Proceedings of the Royal Society of London, Series A, Containing papers of a mathematical and physical character, 115 (772), 700-721, (1927).

[2] J. J. Wang, J. Z. Zhang, Z. Jin, Analysis of a SIR model with bilinear incidence rate, Nonlinear Analysis Real World Applications, 11 (4) 2390–2402, (2010).

[3] Y. Zhao, D. Jiang, The threshold of a stochastic SIS epidemic model with vaccination, Appl. Math. Comput. 243, 718–727, (2014).

[4] A. Settati, A. Lahrouz, A. Assadouq, M. El Fatini, M. El Jarroudi, K. Wang, The impact of nonlinear relapse and reinfection to derive a stochastic threshold for SIRI epidemic model, Chaos, Solitons & Fractals, 137, 109897, https://doi.org/10.1016/j.chaos.2020.109897, (2020).

[5] A. El Haitami, A. Settati, A. Lahrouz, M. El Idrissi, M. El Merzguioui, A stochastic switched SIRI epidemic model integrating nonlinear relapse phenomena, Int. J. Dynam. Control, 1–15, https://doi.org/10.1007/s40435-023-01256-9, (2023).

[6] I. Bouzalmat, M. El Idrissi, A. Settati, A. Lahrouz, Stochastic SIRS epidemic model with perturbation on immunity decay rate, Journal of Applied Mathematics and Computing, 1-26, https://doi.org/10.1007/s12190-023-01937-w, (2023).

[7] M. El Idrissi, B. Harchaoui, A. N. Brahim, I. Bouzalmat, A. Settati, A. Lahrouz, A sufficient condition for extinction and stability of a stochastic SIS model with random perturbation, WSEAS Transactions on Systems, 21, 367-371, (2022).

[8] S. Aznague, M. El Idrissi, A. N. Brahim, B. Harchaoui, S. Boutouil, A. Settati, A. Lahrouz, M. El Merzguioui, J. El Amrani, A Probabilistic SIRI Epidemic Model Incorporating Incidence Capping and Logistic Population Expansion, Appl. Math. Inf. Sci, 17, No. 5, 773–789, http://dx.doi.org/10.18576/amis/170505, (2023).

[9] A. Settati, A. Lahrouz, M. Zahri, A. Tridane, M. El Fatini, H. El Mahjour, M. Seaid, A stochastic threshold to predict extinction and persistence of an epidemic SIRS system with a general incidence rate, Chaos, Solitons & Fractals, 144, 110690, https://doi.org/10.1016/j.chaos.2021.110690, (2021).

[10] T. Harko, F. S. Lobo, M. Mak, Exact analytical solutions of the Susceptible-Infected-Recovered (SIR) epidemic model and the SIR model with equal death and birth rates, Applied Mathematics and Computation, 236, 184-194, (2014).

[11] X. Zhang, Q. Yang, Threshold behavior in a stochastic SVIR model with general incidence rates, Applied Mathematics Letters, 121, 107403, November, (2021).

[12] S. Peng, X. Zhu, Necessary and sufficient condition for comparison theorem of 1-dimensional stochastic differential equations, Stochastic Processes and their Applications, 116, 370–380, (2006).

[13] D. H. Nguyen, G. Yin, C. Zhu, Long-term analysis of a stochastic SIRS model with general incidence rates, SIAM J. Appl. Math, 80, 814–838, (2020).

[14] S. P. Meyn, R. L. Tweedie, Stability of Markovian processes III: Foster-Lyapunov criteria for continuous-time processes, Adv. Appl. Prob, 25, 518–548, (1993).

[15] N. T. Dieu, D. H. Nguyen, N. H. Du, G. Yin, Classification of asymptotic behavior in a stochastic SIR model, SIAM J. Appl. Dyn. Syst, 15, pp. 1062–1084, (2016).

[16] D. Nguyen, N. Nguyen, G. Yin, General nonlinear stochastic systems motivated by chemostat models: Complete characterization of long-time behavior, optimal controls, and applications to wastewater treatment, Stochastic Process, Appl, 130, 4608–4642, (2020).

[17] N. H. Du, N. H. Dang, G. Yin, Conditions for permanence and ergodicity of certain stochastic predator–prey models, J. Appl. Probably, 53, 187–202, (2016).

[18] K. Ichihara, H. Kunita, A classification of the second order degenerate elliptic operators and its probabilistic characterization, Z. Wahrscheinlichkeitstheor, Verwandte Geb, 39, 81–84, (1977).

[19] R. Khasminski, Ergodic properties of recurrent diffusion processes and stabilization of the Cauchy problem for parabolic equations, Theory Probab. Appl, 5, 179–196, (1960).

[20] A. Hening, D. Nguyen, Coexistence and extinction for stochastic Kolmogorov systems, Ann. Appl. Probab, 28, 1893–1942, (2018).

On a Necessary Condition for an Optimal Control Problem in a Parabolic System

Yeşim AKBULUT[1]

[1]*Department of Mathematics, Faculty of Science,*
Atatürk University, Erzurum-TURKEY

ysarac@atauni.edu.tr

ABSTRACT

This study deals with obtaining an optimal solution for the optimal control of the parabolic initial-boundary value problem. We show that the optimal control problem is well-posed and prove that the cost functional is differentiable. In this paper, a variational method for the optimal control problem is suggested. A necessary condition for the optimal solution is given.

1. INTRODUCTION

The control problems of the parabolic equations have attracted many mathematicians since these problems are of paramount importance in heat conduction processes. Bushuyev [15] has controlled the function $f(x)$ in the parabolic problem $u_t + Au = \sigma(x,t)f(x)$ with Dirichlet boundary conditions. Münch and Periago [4] have studied the optimal distribution of the support of the internal null control of minimal L_2-norm for the 1-D heat equation. In [5], the numerical approximation of an optimal control problem for a linear heat equation has been presented. Yu [14] has established the equivalence of minimal time and minimal norm control problems for the semi-linear heat equations. Zheng, Guo and Ali [12] have investigated the stability of the optimization problem for a multidimensional heat equation. Zheng and Yin [13] have studied the optimal time for the time optimal control problem governed by an internally controlled semi-linear heat equation.

There are some studies about the initial control for parabolic problems [7-8,16,20] and for hyperbolic problems [2-3,23]. Klibanov [20] obtained logarithmic stability estimates for the unknown initial condition. Lions [16] has examined the optimal control problem of the initial condition for the parabolic systems from the measured temperature at the final time when the control function belongs to the space L_2. Hao and Oanh worked on the determination of the initial condition in parabolic equations from boundary observations in [7] and from integral observations in [8]. In both studies, the control function belongs to the space L_2.

In this study, we consider a thin rod of length l with an initial temperature $v(x)$, and a heat source $f(x,t)$. We control the initial temperature in a parabolic problem from a given target function $y(x,t)$. Moreover, the initial data is an element of the space H^1. More precisely, we consider the following optimal control problem:

Choose a control $v(x) \in H^1(0,l)$ and a corresponding u such that the pair (v,u) minimizes the functional

$$J_\alpha(v) = \int_0^T \int_0^L [u(x,t;v) - y(x,t)]^2 dx\, dt + \alpha \|v - w\|^2_{H^1(0,l)} \tag{1.1}$$

subject to the parabolic problem:

$$\begin{aligned} &u_t - k u_{xx} = f(x,t), \quad (x,t) \in \Omega := (0,l) \times (0,T] \\ &u(x,0) = v(x), \quad x \in (0,l) \\ &u_x(0,t) = 0, \quad u_x(l,t) = 0, \quad t \in (0,T] \end{aligned} \tag{1.2}$$

where $k > 0$ is a constant, $y \in L_2(\Omega)$ is a given target function, f is a given function and $w \in H^1(0,l)$ is an initial guess for the optimal control.

$J_\alpha(v)$ is called the objective function. With the choice of the functional in (1.1), we mention the observation of $u(x,t;v)$ in $L_2(\Omega)$ for the control $v(x) \in H^1(0,l)$. In (1), $\alpha > 0$ is the parameter of regularization and it can be found by the Tikhonov regularization method [6]. Let admissible set V_{ad} be closed, convex and bounded subset of the space $H^1(0,l)$. We denote by $u(x,t;v)$ the solution of the parabolic problem (1.2), corresponding to the given element v$\in V_{ad}$.

The parabolic boundary value problem (1.2) admits a unique solution $u \in H^{1,1}(\Omega)$ for every $v(x) \in H^1(0,l)$ and $f \in L_2(\Omega)$. This solution holds the following estimate [9,17]:

$$\|u\|^2_{H^{1,1}} \le c_1 \left(\|f\|^2_{L^2(\Omega)} + \|v\|^2_{H^1(0,l)} \right) \tag{1.3}$$

where c_1 is a constant independent from f and v.

2. MAIN RESULTS

In this section, we show the existence and uniqueness of the optimal solution and get the gradient of the cost functional (1.2). Then we give a necessary condition in the integral form for the optimal solution.

Now, we give the difference problem of the optimal control problem (1.1)-(1.2). Let's give the increment Δv to v such that $v + \Delta v \in V_{ad}$ and show the solution of (1.2) corresponding $v + \Delta v$ by $u_\Delta = u(x,t; v + \Delta v)$. Then the function $\Delta u = u_\Delta - u$ will be the solution of the following difference problem:

$$\begin{aligned} &\Delta u_t - k \Delta u_{xx} = 0, \quad (x,t) \in \Omega \\ &\Delta u(x,0) = \Delta v(x), \quad x \in (0,l) \\ &\Delta u_x(0,t) = 0, \quad \Delta u_x(l,t) = 0, \quad t \in (0,T] \end{aligned} \tag{2.1}$$

Lemma 1. *Let Δu be the solution of the problem (2.1). Then the following estimate is valid:*

$$\|\Delta u(x,t)\|^2_{L_2(0,T)} \le c_2 \|\Delta v\|^2_{H^1(0,l)}. \tag{2.2}$$

Here c_2 is independent from Δv.

Theorem 1 (Goebel Theorem). *Let H be a uniformly convex Banach space and the set V be a closed, bounded and convex subset of H. If $\alpha > 0$ and $\beta \geq 1$ are given numbers and the functional $J(v)$ is lower semi continuous and bounded from below on the set V then there is a dense set G of H that the functional*

$$J_\alpha(v) = J(v) + \alpha\|v\|_H^\beta$$

takes its minimum on the set V. If $\beta > 1$ then minimum is unique [19].

We can easily prove that the optimal control problem (1.1)-(1.2) holds the conditions of the Theorem 1. So, we have a unique optimal solution for the optimal control problem (1.1)-(1.2).

Let us introduce the Lagrangian $L(u, v, \psi)$ given by

$$L(u, v, \psi) = J_\alpha(v) + \langle \psi, u_t - ku_{xx} - f(x,t)\rangle_{L_2(\Omega)} \tag{2.3}$$

where the functional $J_\alpha(v)$ is defined by (1.1) and the function $\psi(x,t)$ is the Lagrange multiplier.

Using the $\delta L = 0$ stationarity condition, we have the following adjoint problem:

$$\begin{aligned}
&\psi_t - k\psi_{xx} = 2[u(x,t) - y(x,t)], \quad (x,t) \in \Omega \\
&\psi(x,0) = 0, \quad x \in (0,l) \\
&\psi_x(0,t) = 0, \quad \psi_x(l,t) = 0\,, \quad t \in (0,T]
\end{aligned} \tag{2.4}$$

Now, we investigate the variation of the functional $J_\alpha(v)$. The difference functional $\Delta J_\alpha(v) = J_\alpha(v + \Delta v) - J_\alpha(v)$ is such as

$$\begin{aligned}
\Delta J_\alpha(v) = &\int_0^T \int_0^L [2u(x,t;v) - 2y(x,t)]\Delta u(x,t)dx\,dt \\
&+ \int_0^T \int_0^L [\Delta u(x,t)]^2 dx\,dt + \alpha\int_0^l (2v - 2w + \Delta v)\Delta v dx \\
&+ \alpha\int_0^l (2v' - 2w' + \Delta v')\Delta v' dx\,.
\end{aligned} \tag{2.5}$$

Using the identity between the difference problem and the adjoint problem, the equation (2.5) can be rewritten as follows:

$$\begin{aligned}
\Delta J_\alpha(v) = &\int_0^l -\psi(x,0)\Delta v dx + 2\alpha\int_0^l \big((v - w)\Delta v + (v' - w')\Delta v'\big)dx \\
&+ \int_0^T \int_0^L [\Delta u(x,t)]^2 dx\,dt + \alpha\int_0^l \{(\Delta v)^2 + (\Delta v')^2\}dx
\end{aligned} \tag{2.6}$$

In order to obtain the inner product in the space $H^1(0,l)$ we consider the function $\eta(x)$. Here, the function $\eta(x)$ is the solution to the following second adjoint problem:

$$\begin{aligned}
&\eta'' - \eta = \psi(x,0) \\
&\eta'(0) = \eta'(l) = 0.
\end{aligned} \tag{2.7}$$

We can write that

$$\begin{aligned}\Delta J_\alpha(v) &= \int_0^l \eta' \Delta v' dx + \int_0^l \eta \Delta v dx \\ &+2\alpha \int_0^l \big((v-w)\Delta v + (v'-w')\Delta v'\big) dx \\ &+ \int_0^l [\Delta u(x,T)]^2 \, dx + \alpha \int_0^l \{(\Delta v)^2 + (\Delta v')^2\} dx\,.\end{aligned} \tag{2.8}$$

After some transformation, we have

$$\begin{aligned}\Delta J_\alpha(v) &= \int_0^l \{\eta + 2\alpha(v-w)\} \Delta v dx \\ &+ \int_0^l \{\eta' + 2\alpha(v'-w')\} \Delta v' dx \\ &+ \int_0^l [\Delta u(x,T)]^2 \, dx + \alpha \int_0^l \{(\Delta v)^2 + (\Delta v')^2\} dx\,.\end{aligned} \tag{2.9}$$

From the estimate (2.2), we get the Frechet derivation for the cost functional

$$J'_\alpha(v) = \eta + 2\alpha(v-w). \tag{2.10}$$

Theorem 2. *Let the assumptions of Theorem 1 remain valid and v^* of V_{ad} is the solution of the optimal control problem (1.1)-(1.2). In this case, the following inequality is provided*

$$\langle \eta^* + 2\alpha(v^* - w), v - v^* \rangle_{H^1(0,l)} \geq 0$$

for $\forall v \in V_{ad}$. Here η^* is the solution to the second adjoint problem (2.7) corresponding to the optimal solution v^*[10].

We use the conjugate gradient method that is known to be very successful in linear optimization problems in order to compute a numerical approximation of the optimal control. According to this method the minimizing sequence is set by

$$v_{k+1} = v_k - \beta_k J'_\alpha(v_k), \quad k = 0,1,2,\cdots \tag{2.11}$$

where $v_0 \in V_{ad}$ is a given initial iteration and β_k is a small enough relaxation parameter and assures that

$$J_\alpha(v_{k+1}) < J_\alpha(v_k).$$

Concerning the choice of the parameter β_k, there are several possibilities and these can be found in any optimization books.

REFERENCES

1. D. İskenderov, R.Q. Tagiyev, Q.Y. Yagubov, Optimization Methods, Çaşıoğlu, Baku, 2002.
2. Kowalewski, Optimal control via initial state of an infinite order time delay hyperbolic system, Proceeding of the 18th International Conferences on Process Control, 14-17 June 2011, Tatranska Lomnica, Slovakia.
3. Kowalewski A, Optimal control via initial conditions of a time delay hyperbolic system, The 18th International Conferences on Methods and Models in Automation and Robotics, 22-25 August 2011, Miedzyzdroje, Poland.

4. Münch and F. Periago, Optimal distribution of the internal null control for the one-dimensional heat equation, Journal of Differential Equations, 250 (2011) 95-111.
5. Münch and F. Periago, Numerical approximation of bang-bang controls for the heat equation: an optimal design approach, Sytems and Control Letters, 62 (2013) 643-655.
6. A.N. Tikhonov, and A.A. Samarskii, 1963. Equations of Mathematical Physics. Dover Publications, 765 p, New York.
7. D.N. Hao, N.T.T. Oanh, Determination of the initial condition in parabolic equations from boundary observations, J. Inverse III-Posed Probl., 24(2), 195-220 (2016).
8. D.N. Hào and N.T.N. Oanh, Determination of the initial condition in parabolic equations from integral observations, Inverse Problems in Science and Engineering, 25 (2017), 1138-1167.
9. F. Tröltzsh, Optimal control of partial differential equations: Theory, Methods and Applications, American Mathematical Society, 2010.
10. F. P. Vasilyev, Methods of solving for extremal problems, Nauka, Moskow, 1981
11. G. Ji, C. Martin, Applied Mathematics and Computation 127, 335-345 (2002).
12. G. Zheng, B. Guo and M. M. Ali, Stability of optimal control of heat equation with singular potential, Sytems and Control Letters, 74 (2014) 18-23.
13. G. Zheng, J. Yin, Numerical approximation for a time optimal control problems governed by semi-linear heat equations, Advances in Difference Equations, (2014), 1-7.
14. H. Yu, Equivalence of minimal time and minimal norm control problems for semi linear heat equation, Sytems and Control Letters, 73 (2014) 17-24.
15. Bushuyev, Global uniqueness for inverse parabolic problems with final observation, Inverse Problems. (1995) 11-16.
16. J.L. Lions, 1971, Optimal Control of Systems Governed by Partial Differential Equations, Springer-Verlag, New York.
17. J. Wloka, Partial differential equations, Cambridge University Press, Springer, Berlin, 1971.
18. L. Fardigola, K. Khalina, ESAIM: Control, Optimisation and Calculus of Variations, 28, 1-21 (2022).
19. M. Goebel, On existence of optimal control, Mathematische Nachrichten, 93, 67-73 (1979).
20. M.V. Klibanov, Estimates of initial conditions of parabolic equations and inequalities via lateral Caucht data, Inverse Problems 22, 495-514 (2006).
21. S. Micu, I. Roventa, M. Tucsnak, Journal of Functional Analysis 263, 25-49 (2012).
22. V. Dhamo, F. Tröltzsch, Some aspect of reachability for parabolic boundary control problems with control conctraints, Comput. Optim. Appl. 50, 75-110 (2011).
23. Y. Saraç, Symbolic and Numeric computation of optimal initial velocity in a wave equation, Journal of Computational and Nonlinear Dynamics, 8 (2012), 1-4.

Some Integral Inequalities for Different Classes of Functions via Atangana-Baleanu Fractional Integral Operators

Ali KARAOĞLAN[1], Erhan SET[1] and Ahmet Ocak AKDEMİR[2]

[1]Department of Mathematics, Faculty of Science and Arts,
Ordu University, Ordu-TURKEY
[2]Department of Mathematics, Faculty of Science and Arts,
Ağrı İbrahim Çeçen University, Ağrı-TURKEY

aliblackboy80@hotmail.com
erhanset@yahoo.com
aocakakdemir@gmail.com

ABSTRACT

In this paper, we obtained some integral inequalities via Atangana-Baleanu fractional integral operators for s-convex functions and P-functions using the identity by proved Set et al.. Some of the inequalities proved are reduced to existing inequalities in the literature for some special values of the parameters. And also, the inequalities obtained produce new results for some special values of the parameters.

1. INTRODUCTION

The convex function, whose definition is based on the condition that an inequality is satisfied, has made important contributions to the development of inequality theory. The definition of this function is as follows.

Definition 1.1 *The function* $f:[a,b]\subseteq \mathbb{R}\to\mathbb{R}$, *is said to be convex if the following inequality holds*

$$f(\lambda x+(1-\lambda)y)\le \lambda f(x)+(1-\lambda)f(y) \tag{1.1}$$

for all $x,y\in[a,b]$ and $\lambda\in[0,1]$. We say that f is concave if $(-f)$ is convex.

One of the most important results obtained for convex functions is the Hermite-Hadamard inequality and this inequality has helped researchers to obtain many new results in inequality theory. This important inequality is given follow.

Assume that $f:I\subseteq\mathbb{R}\to\mathbb{R}$ is a convex function defined on the interval I of $\mathbb{R}$ where $a<b$. The following statement;

$$f\left(\frac{a+b}{2}\right)\le \frac{1}{b-a}\int_a^b f(x)dx\le \frac{f(a)+f(b)}{2} \tag{1.2}$$

holds and known as Hermite-Hadamard inequality. Both inequalities hold in the reversed direction if f is concave.

We recall the definitions of $s-$convex function in the second sense.

Definition 1.2 *(see [4],[7]) Let* $0 < s \le 1$. *A function* $f:[0,\infty)\to \mathbb{R}$, *is said to be s-Breckner convex or s-convex in the second sense, if for every* $x, y \in [0,\infty)$ *and* $\alpha, \beta \ge 0$ *with* $\alpha+\beta=1$, *we have*

$$f(\alpha x+\beta y) \le \alpha^s f(x)+\beta^s f(y). \tag{1.3}$$

The set of all s-convex functions in the second sense is denoted by K_s^2.

It can be easily seen that for $s=1$, s-convexity reduces to the ordinary convexity of functions defined on $[0,\infty)$.

In [6] Dragomir and Fitzpatrick proved a variant of Hadamard¡¦s inequality which holds for s-convex functions in the second sense.

Theorem 1.1 *Suppose that* $f:[0,\infty)\to \mathbb{R}$ *is an s-convex function in the second sense, where* $s\in(0,1]$, *and let* $a,b\in[0,\infty)$, $a<b$. *If* $f\in L[a,b]$, *then the following inequalities hold*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{s+1}.$$

The definition of $P-$function is given as follow.

Definition 1.3 *[8] A function* $f: I\subseteq \mathbb{R}\to\mathbb{R}$ *is* $P-$*function or that* f *belongs to the class of* $P(I)$, *if it is nonnegative and, for all* $x,y\in I$ *and* $\lambda\in[0,1]$, *satisfies the following inequality;*

$$f(\lambda x+(1-\lambda)y) \le f(x)+f(y). \tag{1.4}$$

The use of operators defined in fractional analysis in inequality theory has brought a new perspective to the field. Researchers have obtained many new results using these operators and one of the operators defined recently is the Atangana-Baleanu fractional integral operators. Atangana and Baleanu described this fractional integral operator with the help of Laplace transform and convolution theorem as follows.

Definition 1.4 *[2] The fractional integral associate to the new fractional derivative with non-local kernel of a function* $f\in H^1(a,b)$ *as defined:*

$${}_a^{AB}I^{\alpha}\{f(t)\} = \frac{1-\alpha}{B(\alpha)} f(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_a^t f(y)(t-y)^{\alpha-1}dy$$

where $b>a, \alpha\in[0,1]$.

In [1], Abdeljawad and Baleanu introduced right hand side of integral operator as following; The right fractional new integral with Mittag-Leffler kernel of order $\alpha\in[0,1]$ is defined by

$$^{AB}I_b^{\alpha}\{f(t)\}=\frac{1-\alpha}{B(\alpha)}f(t)+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_t^b f(y)(y-t)^{\alpha-1}dy.$$

For more results related to different kinds of fractional operators, we suggest to the interested readers the papers [3, 5, 9, 10, 11, 12, 13, 14].

In this paper, we will denote normalization function as $B(\alpha)$ with $B(0)=B(1)=1$.

The Gamma function $\Gamma(z)$ developed by Euler is usually defined as follow.

Definition 1.5 *[16] Assume that $\Re(z)>0$, the Gamma function is denoted by $\Gamma(z)$ and defined as follow.*

$$\Gamma(z)=\int_0^{\infty}e^{-t}t^{z-1}dt.$$

The definition of Beta function is as follow.

Definition 1.6 *[16] Assume that $\Re(\eta)>0$ and $\Re(\rho)>0$, the Beta function is denoted by $\beta(\eta,\rho)$ and defined as*

$$\beta(\eta,\rho)=\int_0^1 t^{\eta-1}(1-t)^{\rho-1}dt.$$

Set et al. in [15] proved identity that we using to obtained our main results via Atangana-Baleanu fractional integral operators as following.

Lemma 1.1 *$f:[a,b]\to \mathbb{R}$ be differentiable function on (a,b) with $a<b$. Then we have the following identity for Atangana-Baleanu fractional integral operators*

$$\begin{aligned}&{}^{AB}_{\ a}I^{\alpha}\{f(t)\}+{}^{AB}I_b^{\alpha}\{f(t)\}-\frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)}-\frac{2(1-\alpha)f(t)}{B(\alpha)}\\&=\frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1(1-k)^{\alpha}f'(kt+(1-k)a)dk-\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1 k^{\alpha}f'(kb+(1-k)t)dk\end{aligned}$$

where $\alpha\in(0,1], t\in[a,b]$, $k\in[0,1]$, $B(\alpha)>0$ is normalization function and $\Gamma(.)$ is gamma function.

The main purpose of this article is to obtain some integral inequalities that includes the Atangana-Baleanu fractional integral operators for $s-$ convex functions and $P-$functions with the help of the identity by proved Set et al. in [15].

2. MAIN RESULTS

In this part, we obtained some fractional integral inequalities for $s-$convex functions in the second sense and $P-$functions with help of identity by proved Set et al. in [15] as following:

Theorem 2.1 *Let $f:[a,b]\to \mathbb{R}$ be differentiable function on (a,b) with $a<b$ and $f^{'}\in L_1[a,b]$. If $|f^{'}|$ is a $s-$convex function in the second sense, then the following inequality for Atangana-Baleanu fractional integral operators hold*

$$\left| {}_{a}^{AB}I^{\alpha}\{f(t)\}+{}^{AB}I_{b}^{\alpha}\{f(t)\}-\frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)}-\frac{2(1-\alpha)f(t)}{B(\alpha)}\right| \tag{2.1}$$
$$\leq \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\left|f^{'}(t)\right|B(s+1,\alpha+1)+\frac{\left|f^{'}(a)\right|}{\alpha+s+1}\right)$$
$$+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\frac{\left|f^{'}(b)\right|}{\alpha+s+1}+\left|f^{'}(t)\right|B(\alpha+1,s+1)\right)$$

where $t\in[a,b]$, $\alpha\in(0,1]$, $s\in(0,1]$, $B(\alpha)>0$ is normalization function and $\Gamma(.)$ is gamma function.

Proof. By using the identity that is given in Lemma 1.1, we can write

$$\left| {}_{a}^{AB}I^{\alpha}\{f(t)\}+{}^{AB}I_{b}^{\alpha}\{f(t)\}-\frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)}-\frac{2(1-\alpha)f(t)}{B(\alpha)}\right|$$
$$=\left|\frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1(1-k)^{\alpha}f^{'}(kt+(1-k)a)dk-\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1k^{\alpha}f^{'}(kb+(1-k)t)dk\right|$$
$$\leq \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1(1-k)^{\alpha}\left|f^{'}(kt+(1-k)a)\right|dk+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1k^{\alpha}\left|f^{'}(kb+(1-k)t)\right|dk.$$

By using $s-$convexity in the second sense of $|f^{'}|$, we get

$$\left| {}_{a}^{AB}I_{t}^{\alpha}\{f(t)\}+{}_{b}^{AB}I_{t}^{\alpha}\{f(t)\}-\frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)}-\frac{2(1-\alpha)f(t)}{B(\alpha)}\right|$$
$$\leq \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1(1-k)^{\alpha}\left[k^{s}\left|f^{'}(t)\right|+(1-k)^{s}\left|f^{'}(a)\right|\right]dk$$
$$+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1k^{\alpha}\left[k^{s}\left|f^{'}(b)\right|+(1-k)^{s}\left|f^{'}(t)\right|\right]dk$$
$$=\frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1\left[(1-k)^{\alpha}k^{s}\left|f^{'}(t)\right|+(1-k)^{\alpha+s}\left|f^{'}(a)\right|\right]dk$$
$$+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1\left[k^{\alpha+s}\left|f^{'}(b)\right|+k^{\alpha}(1-k)^{s}\left|f^{'}(t)\right|\right]dk$$
$$=\frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\left|f^{'}(t)\right|B(s+1,\alpha+1)+\frac{\left|f^{'}(a)\right|}{\alpha+s+1}\right)$$
$$+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\frac{\left|f^{'}(b)\right|}{\alpha+s+1}+\left|f^{'}(t)\right|B(\alpha+1,s+1)\right)$$

and the proof is completed.

Corollary 2.1 *In Theorem 2.1, if we choose* $t=\frac{a+b}{2}$ *we obtain*

$$\left| {}_{a}^{AB}I^{\alpha} f\left(\frac{a+b}{2}\right) + {}^{AB}I_{b}^{\alpha} f\left(\frac{a+b}{2}\right) - \frac{(b-a)^{\alpha}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}[f(a)+f(b)] - \frac{2(1-\alpha)f\left(\frac{a+b}{2}\right)}{B(\alpha)} \right|$$

$$\leq \frac{(b-a)^{\alpha+1}}{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}\left(2\left|f'(t)\right|B(s+1,\alpha+1)+\frac{\left|f'(a)\right|+\left|f'(b)\right|}{\alpha+s+1}\right).$$

Remark 2.1 *In Theorem 2.1, if we choose* $s=1$*, the inequality (2.1) reduces to the inequality in Theorem 2.2 in [15].*

Theorem 2.2 *Let* $f:[a,b]\to \mathrm{R}$ *be differentiable function on* (a,b) *with* $a<b$ *and* $f' \in L_1[a,b]$. *If* $|f'|^q$ *is a* $s-$*convex function in the second sense, then the following inequality for Atangana-Baleanu fractional integral operators hold*

$$\left| {}_{a}^{AB}I^{\alpha}\{f(t)\} + {}^{AB}I_{b}^{\alpha}\{f(t)\} - \frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)f(t)}{B(\alpha)} \right| \tag{2.2}$$

$$\leq \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{\left|f'(a)\right|^{q}+\left|f'(t)\right|^{q}}{s+1}\right)^{\frac{1}{q}}$$

$$+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{\left|f'(t)\right|^{q}+\left|f'(b)\right|^{q}}{s+1}\right)^{\frac{1}{q}}$$

where $p^{-1}+q^{-1}=1$, $t\in[a,b]$, $\alpha\in(0,1]$, $s\in(0,1]$, $q>1$, $B(\alpha)>0$ is normalization function and $\Gamma(.)$ is gamma function.

Proof. By using Lemma 1.1, we have

$$\left| {}_{a}^{AB}I^{\alpha}\{f(t)\} + {}^{AB}I_{b}^{\alpha}\{f(t)\} - \frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)f(t)}{B(\alpha)} \right|$$

$$\leq \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1(1-k)^{\alpha}\left|f'(kt+(1-k)a)\right|dk+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1 k^{\alpha}\left|f'(kb+(1-k)t)\right|dk.$$

By applying Hölder inequality, we have

$$\left| {}_{a}^{AB}I^{\alpha}\{f(t)\} + {}^{AB}I_{b}^{\alpha}\{f(t)\} - \frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)f(t)}{B(\alpha)} \right|$$

$$\leq \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1 (1-k)^{\alpha p}dk\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'(kt+(1-k)a)\right|^q dk\right)^{\frac{1}{q}}\right]$$

$$+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1 k^{\alpha p}dk\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'(kb+(1-k)t)\right|^q dk\right)^{\frac{1}{q}}\right].$$

By using $s-$convexity in the second sense of $|f'|^q$, we obtain

$$\left|{}^{AB}_{a}I^{\alpha}\{f(t)\}+{}^{AB}I_b^{\alpha}\{f(t)\}-\frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)}-\frac{2(1-\alpha)f(t)}{B(\alpha)}\right|$$

$$\leq \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1 (1-k)^{\alpha p}dk\right)^{\frac{1}{p}}\left(\int_0^1 \left[k^s\left|f'(t)\right|^q+(1-k)^s\left|f'(a)\right|^q\right]dk\right)^{\frac{1}{q}}\right]$$

$$+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1 k^{\alpha p}dk\right)^{\frac{1}{p}}\left(\int_0^1 \left[k^s\left|f'(b)\right|^q+(1-k)^s\left|f'(t)\right|^q\right]dk\right)^{\frac{1}{q}}\right]$$

$$=\frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{\left|f'(t)\right|^q}{s+1}+\frac{\left|f'(a)\right|^q}{s+1}\right)^{\frac{1}{q}}$$

$$+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{\left|f'(b)\right|^q}{s+1}+\frac{\left|f'(t)\right|^q}{s+1}\right)^{\frac{1}{q}}.$$

So, the proof is completed.

Corollary 2.2 *In Theorem 2.2, if we choose* $t=\frac{a+b}{2}$ *we obtain*

$$\left|{}^{AB}_{a}I^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{AB}I_b^{\alpha}f\left(\frac{a+b}{2}\right)-\frac{(b-a)^{\alpha}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}[f(a)+f(b)]-\frac{2(1-\alpha)f\left(\frac{a+b}{2}\right)}{B(\alpha)}\right|$$

$$\leq \frac{(b-a)^{\alpha+1}}{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[\left(\frac{\left|f'(a)\right|^q+\left|f'(\frac{a+b}{2})\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(\frac{a+b}{2})\right|^q+\left|f'(b)\right|^q}{s+1}\right)^{\frac{1}{q}}\right].$$

Remark 2.2 *In Theorem 2.2, if we choose* $s=1$, *the inequality (2.2) reduces to the inequality in Theorem 2.5 in [15].*

Theorem 2.3 *Let $f:[a,b]\to \mathbb{R}$ be differentiable function on (a,b) with $a<b$ and $f' \in L_1[a,b]$. If $|f'|^q$ is a $s-$convex function in the second sense, then the following inequality for Atangana-Baleanu fractional integral operators hold*

$$\left| {}_{a}^{AB}I^{\alpha}\{f(t)\} + {}^{AB}I_b^{\alpha}\{f(t)\} - \frac{(t-a)^{\alpha} f(a) + (b-t)^{\alpha} f(b)}{B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha) f(t)}{B(\alpha)} \right| \tag{2.3}$$

$$\le \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)} \left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}} \left(\left|f'(t)\right|^q B(s+1,\alpha+1) + \frac{\left|f'(a)\right|^q}{\alpha+s+1} \right)^{\frac{1}{q}}$$

$$+ \frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)} \left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}} \left(\frac{\left|f'(b)\right|^q}{\alpha+s+1} + \left|f'(t)\right|^q B(\alpha+1,s+1) \right)^{\frac{1}{q}}$$

where $t\in[a,b]$, $\alpha\in(0,1]$, $s\in(0,1]$, $q\ge 1$, $B(\alpha)>0$ is normalization function and $\Gamma(.)$ is gamma function.

Proof. By using the identity that is given in Lemma 1.1, we can write

$$\left| {}_{a}^{AB}I^{\alpha}\{f(t)\} + {}^{AB}I_b^{\alpha}\{f(t)\} - \frac{(t-a)^{\alpha} f(a) + (b-t)^{\alpha} f(b)}{B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha) f(t)}{B(\alpha)} \right|$$

$$\le \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)} \int_0^1 (1-k)^{\alpha} \left|f'(kt+(1-k)a)\right| dk + \frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)} \int_0^1 k^{\alpha} \left|f'(kb+(1-k)t)\right| dk.$$

By applying power mean inequality, we get

$$\left| {}_{a}^{AB}I^{\alpha}\{f(t)\} + {}^{AB}I_b^{\alpha}\{f(t)\} - \frac{(t-a)^{\alpha} f(a) + (b-t)^{\alpha} f(b)}{B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha) f(t)}{B(\alpha)} \right|$$

$$\le \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)} \left[\left(\int_0^1 (1-k)^{\alpha} dk \right)^{1-\frac{1}{q}} \left(\int_0^1 (1-k)^{\alpha} \left|f'(kt+(1-k)a)\right|^q dk \right)^{\frac{1}{q}} \right]$$

$$+ \frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)} \left[\left(\int_0^1 k^{\alpha} dk \right)^{1-\frac{1}{q}} \left(\int_0^1 k^{\alpha} \left|f'(kb+(1-k)t)\right|^q dk \right)^{\frac{1}{q}} \right].$$

By using $s-$convexity in the second sense of $|f'|^q$, we obtain

$$\left| {}_{a}^{AB}I^{\alpha}\{f(t)\} + {}^{AB}I_b^{\alpha}\{f(t)\} - \frac{(t-a)^{\alpha} f(a) + (b-t)^{\alpha} f(b)}{B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha) f(t)}{B(\alpha)} \right|$$

$$\le \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)} \left[\left(\int_0^1 (1-k)^{\alpha} dk \right)^{1-\frac{1}{q}} \left(\int_0^1 (1-k)^{\alpha} \left[k^s \left|f'(t)\right|^q + (1-k)^s \left|f'(a)\right|^q \right] dk \right)^{\frac{1}{q}} \right]$$

$$+ \frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)} \left[\left(\int_0^1 k^{\alpha} dk \right)^{1-\frac{1}{q}} \left(\int_0^1 k^{\alpha} \left[k^s \left|f'(b)\right|^q + (1-k)^s \left|f'(t)\right|^q \right] dk \right)^{\frac{1}{q}} \right]$$

$$= \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1 (1-k)^{\alpha} dk\right)^{1-\frac{1}{q}}\left(\int_0^1\left[(1-k)^{\alpha} k^s \left|f'(t)\right|^q + (1-k)^{\alpha+s}\left|f'(a)\right|^q\right]dk\right)^{\frac{1}{q}}\right]$$

$$+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1 k^{\alpha} dk\right)^{1-\frac{1}{q}}\left(\int_0^1\left[k^{\alpha+s}\left|f'(b)\right|^q + k^{\alpha}(1-k)^s\left|f'(t)\right|^q\right]dk\right)^{\frac{1}{q}}\right]$$

$$= \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\left|f'(t)\right|^q B(s+1,\alpha+1) + \frac{\left|f'(a)\right|^q}{\alpha+s+1}\right)^{\frac{1}{q}}$$

$$+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\frac{\left|f'(b)\right|^q}{\alpha+s+1} + \left|f'(t)\right|^q B(\alpha+1,s+1)\right)^{\frac{1}{q}}$$

and the proof is completed.

Corollary 2.3 *In Theorem 2.3, if we choose* $t=\frac{a+b}{2}$ *we obtain*

$$\left|{}^{AB}_{\ a}I^{\alpha} f\left(\frac{a+b}{2}\right) + {}^{AB}I_b^{\alpha} f\left(\frac{a+b}{2}\right) - \frac{(b-a)^{\alpha}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}[f(a)+f(b)] - \frac{2(1-\alpha)f\left(\frac{a+b}{2}\right)}{B(\alpha)}\right|$$

$$\leq \frac{(b-a)^{\alpha+1}}{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}[\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q B(s+1,\alpha+1) + \frac{\left|f'(a)\right|^q}{\alpha+s+1}\right)^{\frac{1}{q}}$$

$$+\left(\frac{\left|f'(b)\right|^q}{\alpha+s+1} + \left|f'\left(\frac{a+b}{2}\right)\right|^q B(\alpha+1,s+1)\right)^{\frac{1}{q}}].$$

Remark 2.3 *In Theorem 2.3, if we choose* $s=1$, *the inequality (2.3) reduces to the inequality in Theorem 2.10 in [15].*

Theorem 2.4 *Let* $f:[a,b]\to \mathbb{R}$ *be differentiable function on* (a,b) *with* $a<b$ *and* $f' \in L_1[a,b]$. *If* $|f'|^q$ *is a* $s-$*convex function in the second sense, then the following inequality for Atangana-Baleanu fractional integral operators hold*

$$\left|{}^{AB}_{\ a}I^{\alpha}\{f(t)\} + {}^{AB}I_b^{\alpha}\{f(t)\} - \frac{(t-a)^{\alpha} f(a) + (b-t)^{\alpha} f(b)}{B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)f(t)}{B(\alpha)}\right| \tag{2.4}$$

$$\leq \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\frac{1}{p(\alpha p+1)}+\frac{\left[\left|f'(a)\right|^q+\left|f'(t)\right|^q\right]}{q(s+1)}\right)$$

$$+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\frac{1}{p(\alpha p+1)}+\frac{\left[\left|f'(t)\right|^q+\left|f'(b)\right|^q\right]}{q(s+1)}\right)$$

where $p^{-1}+q^{-1}=1$, $t\in[a,b]$, $\alpha\in(0,1]$, $s\in(0,1]$ $q>1$, $B(\alpha)>0$ is normalization function and $\Gamma(.)$ is gamma function.

Proof. By using Lemma 1.1, we have

$$\left|{}^{AB}_{a}I^{\alpha}\{f(t)\}+{}^{AB}I^{\alpha}_{b}\{f(t)\}-\frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)}-\frac{2(1-\alpha)f(t)}{B(\alpha)}\right|$$

$$\leq \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1(1-k)^{\alpha}\left|f'(kt+(1-k)a)\right|dk+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1 k^{\alpha}\left|f'(kb+(1-k)t)\right|dk.$$

By applying the Young inequality, we get

$$\left|{}^{AB}_{a}I^{\alpha}\{f(t)\}+{}^{AB}I^{\alpha}_{b}\{f(t)\}-\frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)}-\frac{2(1-\alpha)f(t)}{B(\alpha)}\right|$$

$$\leq \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left[\frac{1}{p}\int_0^1(1-k)^{\alpha p}dk+\frac{1}{q}\int_0^1\left|f'(kt+(1-k)a)\right|^q dk\right]$$

$$+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left[\frac{1}{p}\int_0^1 k^{\alpha p}dk+\frac{1}{q}\int_0^1\left|f'(kb+(1-k)t)\right|^q dk\right].$$

By using $s-$convexity in the second sense of $|f'|^q$, we get

$$\left|{}^{AB}_{a}I^{\alpha}\{f(t)\}+{}^{AB}I^{\alpha}_{b}\{f(t)\}-\frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)}-\frac{2(1-\alpha)f(t)}{B(\alpha)}\right|$$

$$\leq \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left[\frac{1}{p}\int_0^1(1-k)^{\alpha p}dk+\frac{1}{q}\int_0^1\left[k^s\left|f'(t)\right|^q+(1-k)^s\left|f'(a)\right|^q\right]dk\right]$$

$$+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left[\frac{1}{p}\int_0^1 k^{\alpha p}dk+\frac{1}{q}\int_0^1\left[k^s\left|f'(b)\right|^q+(1-k)^s\left|f'(a)\right|^q\right]dk\right]$$

$$=\frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\frac{1}{p(\alpha p+1)}+\frac{\left[\left|f'(a)\right|^q+\left|f'(t)\right|^q\right]}{q(s+1)}\right)$$

$$+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\frac{1}{p(\alpha p+1)}+\frac{\left[\left|f'(t)\right|^q+\left|f'(b)\right|^q\right]}{q(s+1)}\right)$$

and the proof is completed.

Corollary 2.4 *In Theorem 2.4, if we choose* $t = \dfrac{a+b}{2}$ *we obtain*

$$\left| {}^{AB}_{a}I^{\alpha} f\left(\frac{a+b}{2}\right) + {}^{AB}I^{\alpha}_{b} f\left(\frac{a+b}{2}\right) - \frac{(b-a)^{\alpha}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}[f(a)+f(b)] - \frac{2(1-\alpha)f\left(\frac{a+b}{2}\right)}{B(\alpha)} \right|$$

$$\leq \frac{(b-a)^{\alpha+1}}{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}\left(\frac{2}{p(\alpha p+1)} + \frac{2\left|f'(\frac{a+b}{2})\right|^{q} + \left|f'(a)\right|^{q} + \left|f'(b)\right|^{q}}{q(s+1)} \right).$$

Remark 2.4 *In Theorem 2.4, if we choose* $s=1$, *the inequality (2.4) reduces to the inequality in Theorem 2.8 in [15].*

Theorem 2.5 *Let* $f:[a,b]\to \mathbb{R}$ *be differentiable function on* (a,b) *with* $a<b$ *and* $f' \in L_1[a,b]$. *If* $|f'|$ *is a P-function, then the following inequality for Atangana-Baleanu fractional integral operators hold*

$$\left| {}^{AB}_{a}I^{\alpha}\{f(t)\} + {}^{AB}I^{\alpha}_{b}\{f(t)\} - \frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)f(t)}{B(\alpha)} \right|$$

$$\leq \frac{(t-a)^{\alpha+1}\left[\left|f'(t)\right|+\left|f'(a)\right|\right]+(b-t)^{\alpha+1}\left[\left|f'(b)\right|+\left|f'(t)\right|\right]}{B(\alpha)\Gamma(\alpha)(\alpha+1)}$$

where $t\in[a,b]$, $\alpha\in(0,1]$, $B(\alpha)>0$ is normalization function and $\Gamma(.)$ is gamma function.

Proof. By using the identity that is given in Lemma 1.1, we can write

$$\left| {}^{AB}_{a}I^{\alpha}\{f(t)\} + {}^{AB}I^{\alpha}_{b}\{f(t)\} - \frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)f(t)}{B(\alpha)} \right|$$

$$= \left| \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1 (1-k)^{\alpha} f'(kt+(1-k)a)dk - \frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1 k^{\alpha} f'(kb+(1-k)t)dk \right|$$

$$\leq \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1 (1-k)^{\alpha}\left|f'(kt+(1-k)a)\right|dk + \frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1 k^{\alpha}\left|f'(kb+(1-k)t)\right|dk.$$

Since $|f'|$ is $P-$function, we get

$$\left| {}^{AB}_{a}I^{\alpha}_{t}\{f(t)\} + {}^{AB}_{b}I^{\alpha}_{t}\{f(t)\} - \frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)f(t)}{B(\alpha)} \right|$$

$$\leq \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1 (1-k)^{\alpha}\left[\left|f'(t)\right|+\left|f'(a)\right|\right]dk + \frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1 k^{\alpha}\left[\left|f'(b)\right|+\left|f'(t)\right|\right]dk$$

$$=\frac{(t-a)^{\alpha+1}\left[\left|f'(t)\right|+\left|f'(a)\right|\right]+(b-t)^{\alpha+1}\left[\left|f'(b)\right|+\left|f'(t)\right|\right]}{B(\alpha)\Gamma(\alpha)(\alpha+1)}$$

and the proof is completed.

Corollary 2.5 *In Theorem 2.5, if we choose* $t=\frac{a+b}{2}$ *we obtain*

$$\left|{}_{a}^{AB}I^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{AB}I_{b}^{\alpha}f\left(\frac{a+b}{2}\right)-\frac{(b-a)^{\alpha}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}[f(a)+f(b)]-\frac{2(1-\alpha)f\left(\frac{a+b}{2}\right)}{B(\alpha)}\right|$$

$$\leq\frac{(b-a)^{\alpha+1}}{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}\left[\frac{2\left|f'(\frac{a+b}{2})\right|+\left|f'(a)\right|+\left|f'(b)\right|}{\alpha+1}\right].$$

Corollary 2.6 *In Theorem 2.5, if we choose* $\alpha=1$ *we obtain*

$$\left|\int_{a}^{b}f(x)dx-\left[(t-a)f(a)+(b-t)f(b)\right]\right|$$

$$\leq(t-a)^{2}\left[\frac{\left|f'(t)\right|+\left|f'(a)\right|}{2}\right]+(b-t)^{2}\left[\frac{\left|f'(b)\right|+\left|f'(t)\right|}{2}\right].$$

Theorem 2.6 *Let* $f:[a,b]\to\mathrm{R}$ *be differentiable function on* (a,b) *with* $a<b$ *and* $f'\in L_{1}[a,b]$. *If* $|f'|^{q}$ *is a P-function, then the following inequality for Atangana-Baleanu fractional integral operators hold*

$$\left|{}_{a}^{AB}I^{\alpha}\{f(t)\}+{}^{AB}I_{b}^{\alpha}\{f(t)\}-\frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)}-\frac{2(1-\alpha)f(t)}{B(\alpha)}\right|$$

$$\leq\frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\left|f'(t)\right|^{q}+\left|f'(a)\right|^{q}\right)^{\frac{1}{q}}$$

$$+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\left|f'(b)\right|^{q}+\left|f'(t)\right|^{q}\right)^{\frac{1}{q}}$$

where $p^{-1}+q^{-1}=1$, $t\in[a,b]$, $\alpha\in(0,1]$, $q>1$, $B(\alpha)>0$ is normalization function and $\Gamma(.)$ is gamma function.

Proof. By using Lemma 1.1, we have

$$\left|{}_{a}^{AB}I^{\alpha}\{f(t)\}+{}^{AB}I_{b}^{\alpha}\{f(t)\}-\frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)}-\frac{2(1-\alpha)f(t)}{B(\alpha)}\right|$$

$$\leq \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1 (1-k)^{\alpha}\left|f'(kt+(1-k)a)\right|dk + \frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1 k^{\alpha}\left|f'(kb+(1-k)t)\right|dk.$$

By applying Hölder inequality, we have

$$\left|{}_{a}^{AB}I^{\alpha}\{f(t)\}+{}^{AB}I_{b}^{\alpha}\{f(t)\}-\frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)}-\frac{2(1-\alpha)f(t)}{B(\alpha)}\right|$$

$$\leq \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1 (1-k)^{\alpha p}dk\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'(kt+(1-k)a)\right|^q dk\right)^{\frac{1}{q}}\right]$$

$$+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1 k^{\alpha p}dk\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'(kb+(1-k)t)\right|^q dk\right)^{\frac{1}{q}}\right].$$

Since $|f'|^q$ is a $P-$function, we obtain

$$\left|{}_{a}^{AB}I^{\alpha}\{f(t)\}+{}^{AB}I_{b}^{\alpha}\{f(t)\}-\frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)}-\frac{2(1-\alpha)f(t)}{B(\alpha)}\right|$$

$$\leq \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1 (1-k)^{\alpha p}dk\right)^{\frac{1}{p}}\left(\int_0^1 \left[\left|f'(t)\right|^q+\left|f'(a)\right|^q\right]dk\right)^{\frac{1}{q}}\right]$$

$$+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1 k^{\alpha p}dk\right)^{\frac{1}{p}}\left(\int_0^1 \left[\left|f'(b)\right|^q+\left|f'(t)\right|^q\right]dk\right)^{\frac{1}{q}}\right]$$

$$=\frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\left|f'(t)\right|^q+\left|f'(a)\right|^q\right)^{\frac{1}{q}}$$

$$+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\left|f'(b)\right|^q+\left|f'(t)\right|^q\right)^{\frac{1}{q}}.$$

So the proof is completed.

Corollary 2.7 *In Theorem 2.6, if we choose* $t=\dfrac{a+b}{2}$ *we obtain*

$$\left|{}_{a}^{AB}I^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{AB}I_{b}^{\alpha}f\left(\frac{a+b}{2}\right)-\frac{(b-a)^{\alpha}}{2^{\alpha}B(\alpha)\Gamma(\alpha)}[f(a)+f(b)]-\frac{2(1-\alpha)f\left(\frac{a+b}{2}\right)}{B(\alpha)}\right|$$

$$\leq \frac{(b-a)^{\alpha+1}}{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(a)\right|^q\right)^{\frac{1}{q}}+\left(\left|f'(b)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right].$$

Corollary 2.8 *In Theorem 2.6, if we choose* $\alpha=1$ *we obtain*

$$\left|\int_a^b f(x)dx - [(t-a)f(a)+(b-t)f(b)]\right|$$
$$\leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}}(t-a)^2\left(\left|f'(t)\right|^q+\left|f'(a)\right|^q\right)^{\frac{1}{q}}$$
$$+\left(\frac{1}{p+1}\right)^{\frac{1}{p}}(b-t)^2\left(\left|f'(b)\right|^q+\left|f'(t)\right|^q\right)^{\frac{1}{q}}.$$

Theorem 2.7 *Let* $f:[a,b]\to \mathbb{R}$ *be differentiable function on* (a,b) *with* $a<b$ *and* $f' \in L_1[a,b]$. *If* $|f'|^q$ *is a* $P-$*function, then the following inequality for Atangana-Baleanu fractional integral operators hold*

$$\left| {}^{AB}_{\ a}I^{\alpha}\{f(t)\} + {}^{AB}I^{\alpha}_{b}\{f(t)\} - \frac{(t-a)^{\alpha} f(a)+(b-t)^{\alpha} f(b)}{B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)f(t)}{B(\alpha)}\right|$$
$$\leq \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\frac{\left|f'(t)\right|^q+\left|f'(a)\right|^q}{\alpha+1}\right)^{\frac{1}{q}}$$
$$+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\frac{\left|f'(b)\right|^q+\left|f'(t)\right|^q}{\alpha+1}\right)^{\frac{1}{q}}$$

where $t\in[a,b]$, $\alpha\in(0,1]$, $q\geq 1$, $B(\alpha)>0$ is normalization function and $\Gamma(.)$ is gamma function.

Proof. By using the identity that is given in Lemma 1.1, we can write

$$\left| {}^{AB}_{\ a}I^{\alpha}\{f(t)\} + {}^{AB}I^{\alpha}_{b}\{f(t)\} - \frac{(t-a)^{\alpha} f(a)+(b-t)^{\alpha} f(b)}{B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)f(t)}{B(\alpha)}\right|$$
$$\leq \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1(1-k)^{\alpha}\left|f'(kt+(1-k)a)\right|dk + \frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1 k^{\alpha}\left|f'(kb+(1-k)t)\right|dk.$$

By applying power mean inequality, we get

$$\left| {}^{AB}_{\ a}I^{\alpha}\{f(t)\} + {}^{AB}I^{\alpha}_{b}\{f(t)\} - \frac{(t-a)^{\alpha} f(a)+(b-t)^{\alpha} f(b)}{B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha)f(t)}{B(\alpha)}\right|$$
$$\leq \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1(1-k)^{\alpha}dk\right)^{1-\frac{1}{q}}\left(\int_0^1(1-k)^{\alpha}\left|f'(kt+(1-k)a)\right|^q dk\right)^{\frac{1}{q}}\right]$$
$$+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1 k^{\alpha}dk\right)^{1-\frac{1}{q}}\left(\int_0^1 k^{\alpha}\left|f'(kb+(1-k)t)\right|^q dk\right)^{\frac{1}{q}}\right].$$

Since $|f'|^q$ is $P-$ function, we obtain

$$\left| {}_{a}^{AB}I^{\alpha}\{f(t)\} + {}^{AB}I_{b}^{\alpha}\{f(t)\} - \frac{(t-a)^{\alpha} f(a) + (b-t)^{\alpha} f(b)}{B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha) f(t)}{B(\alpha)} \right|$$

$$\leq \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)} \left[\left(\int_0^1 (1-k)^{\alpha} dk \right)^{1-\frac{1}{q}} \left(\int_0^1 (1-k)^{\alpha} \left[\left| f'(t) \right|^q + \left| f'(a) \right|^q \right] dk \right)^{\frac{1}{q}} \right]$$

$$+ \frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)} \left[\left(\int_0^1 k^{\alpha} dk \right)^{1-\frac{1}{q}} \left(\int_0^1 k^{\alpha} \left[\left| f'(b) \right|^q + \left| f'(t) \right|^q \right] dk \right)^{\frac{1}{q}} \right]$$

$$= \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)} \left(\frac{1}{\alpha+1} \right)^{1-\frac{1}{q}} \left(\frac{\left| f'(t) \right|^q + \left| f'(a) \right|^q}{\alpha+1} \right)^{\frac{1}{q}}$$

$$+ \frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)} \left(\frac{1}{\alpha+1} \right)^{1-\frac{1}{q}} \left(\frac{\left| f'(b) \right|^q + \left| f'(t) \right|^q}{\alpha+1} \right)^{\frac{1}{q}}.$$

So, the proof is completed.

Corollary 2.9 *In Theorem 2.7, if we choose* $t = \dfrac{a+b}{2}$ *we obtain*

$$\left| {}_{a}^{AB}I^{\alpha} f\left(\frac{a+b}{2} \right) + {}^{AB}I_{b}^{\alpha} f\left(\frac{a+b}{2} \right) - \frac{(b-a)^{\alpha}}{2^{\alpha} B(\alpha)\Gamma(\alpha)} [f(a) + f(b)] - \frac{2(1-\alpha) f\left(\frac{a+b}{2} \right)}{B(\alpha)} \right|$$

$$\leq \frac{(b-a)^{\alpha+1}}{2^{\alpha+1} B(\alpha)\Gamma(\alpha)} \left(\frac{1}{\alpha+1} \right)^{1-\frac{1}{q}} [\left(\frac{\left| f'(\frac{a+b}{2}) \right|^q + \left| f'(a) \right|^q}{\alpha+1} \right)^{\frac{1}{q}} + \left(\frac{\left| f'(b) \right|^q + \left| f'(\frac{a+b}{2}) \right|^q}{\alpha+1} \right)^{\frac{1}{q}}].$$

Corollary 2.10 *In Theorem 2.7, if we choose* $\alpha = 1$ *we obtain*

$$\left| \int_a^b f(x)dx - [(t-a) f(a) + (b-t) f(b)] \right|$$

$$\leq \left(\frac{1}{2} \right)^{1-\frac{1}{q}} (t-a)^2 \left(\frac{\left| f'(t) \right|^q + \left| f'(a) \right|^q}{2} \right)^{\frac{1}{q}}$$

$$+ \left(\frac{1}{2} \right)^{1-\frac{1}{q}} (b-t)^2 \left(\frac{\left| f'(b) \right|^q + \left| f'(t) \right|^q}{2} \right)^{\frac{1}{q}}.$$

Theorem 2.8 *Let* $f:[a,b]\to \mathrm{R}$ *be differentiable function on* (a,b) *with* $a<b$ *and* $f^{'}\in L_1[a,b]$. *If* $|f^{'}|^q$ *is a* $P-$*function, then the following inequality for Atangana-Baleanu fractional integral operators hold*

$$\left| {}_{a}^{AB}I^{\alpha}\{f(t)\}+{}^{AB}I_{b}^{\alpha}\{f(t)\}-\frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)}-\frac{2(1-\alpha)f(t)}{B(\alpha)}\right|$$
$$\leq \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\frac{1}{p(\alpha p+1)}+\frac{\left[\left|f^{'}(t)\right|^{q}+\left|f^{'}(a)\right|^{q}\right]}{q}\right)$$
$$+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\frac{1}{p(\alpha p+1)}+\frac{\left[\left|f^{'}(b)\right|^{q}+\left|f^{'}(t)\right|^{q}\right]}{q}\right)$$

where $p^{-1}+q^{-1}=1$, $t\in[a,b]$, $\alpha\in(0,1]$, $q>1$, $B(\alpha)>0$ is normalization function and $\Gamma(.)$ is gamma function.

Proof. By using Lemma 1.1, we have

$$\left| {}_{a}^{AB}I^{\alpha}\{f(t)\}+{}^{AB}I_{b}^{\alpha}\{f(t)\}-\frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)}-\frac{2(1-\alpha)f(t)}{B(\alpha)}\right|$$
$$\leq \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1(1-k)^{\alpha}\left|f^{'}(kt+(1-k)a)\right|dk+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\int_0^1 k^{\alpha}\left|f^{'}(kb+(1-k)t)\right|dk.$$

By using the Young inequality, we have

$$\left| {}_{a}^{AB}I^{\alpha}\{f(t)\}+{}^{AB}I_{b}^{\alpha}\{f(t)\}-\frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)}-\frac{2(1-\alpha)f(t)}{B(\alpha)}\right|$$
$$\leq \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left[\frac{1}{p}\int_0^1(1-k)^{\alpha p}dk+\frac{1}{q}\int_0^1\left|f^{'}(kt+(1-k)a)\right|^{q}dk\right]$$
$$+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left[\frac{1}{p}\int_0^1 k^{\alpha p}dk+\frac{1}{q}\int_0^1\left|f^{'}(kb+(1-k)t)\right|^{q}dk\right].$$

Since $|f^{'}|^q$ is $P-$function, we get

$$\left| {}_{a}^{AB}I^{\alpha}\{f(t)\}+{}^{AB}I_{b}^{\alpha}\{f(t)\}-\frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)}-\frac{2(1-\alpha)f(t)}{B(\alpha)}\right|$$
$$\leq \frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left[\frac{1}{p}\int_0^1(1-k)^{\alpha p}dk+\frac{1}{q}\int_0^1\left[\left|f^{'}(t)\right|^{q}+\left|f^{'}(a)\right|^{q}\right]dk\right]$$
$$+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left[\frac{1}{p}\int_0^1 k^{\alpha p}dk+\frac{1}{q}\int_0^1\left[\left|f^{'}(b)\right|^{q}+\left|f^{'}(a)\right|^{q}\right]dk\right]$$

$$=\frac{(t-a)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\frac{1}{p(\alpha p+1)}+\frac{\left[\left|f'(t)\right|^q+\left|f'(a)\right|^q\right]}{q}\right)$$

$$+\frac{(b-t)^{\alpha+1}}{B(\alpha)\Gamma(\alpha)}\left(\frac{1}{p(\alpha p+1)}+\frac{\left[\left|f'(b)\right|^q+\left|f'(t)\right|^q\right]}{q}\right)$$

and the proof is completed.

Corollary 2.11 *In Theorem 2.8, if we choose* $t=\frac{a+b}{2}$ *we obtain*

$$\left|{}_a^{AB}I^\alpha f\left(\frac{a+b}{2}\right)+{}^{AB}I_b^\alpha f\left(\frac{a+b}{2}\right)-\frac{(b-a)^\alpha}{2^\alpha B(\alpha)\Gamma(\alpha)}[f(a)+f(b)]-\frac{2(1-\alpha)f\left(\frac{a+b}{2}\right)}{B(\alpha)}\right|$$

$$\leq\frac{(b-a)^{\alpha+1}}{2^{\alpha+1}B(\alpha)\Gamma(\alpha)}\left(\frac{2}{p(\alpha p+1)}+\frac{2\left|f'(\frac{a+b}{2})\right|^q+\left|f'(a)\right|^q+\left|f'(b)\right|^q}{q}\right).$$

Corollary 2.12 *In Theorem 2.8, if we choose* $\alpha=1$ *we obtain*

$$\left|\int_a^b f(x)dx-[(t-a)f(a)+(b-t)f(b)]\right|$$

$$\leq(t-a)^2\left(\frac{1}{p(p+1)}+\frac{\left|f'(t)\right|^q+\left|f'(a)\right|^q}{q}\right)$$

$$+(b-t)^2\left(\frac{1}{p(p+1)}+\frac{\left|f'(b)\right|^q+\left|f'(t)\right|^q}{q}\right).$$

REFERENCES

1. T. Abdeljawad and D. Baleanu, Integration by parts and its applications of a new nonlocal fractional derivative with Mittag-Leffler nonsingular kernel, J. Nonlinear Sci. Appl., 10 (2017), 1098-1107.
2. A. Atangana and D. Baleanu, New fractional derivatices with non-local and non-singular kernel, Theory and Application to Heat Transfer Model, Thermal Science, Year 2016, Vol. 20, No. 2, pp. 763-769.
3. M.U. Awan, M.A. Noor, M.V. Mihai, K.I. Noor, Conformable fractional Hermite-Hadamard inequalities via preinvex functions. Tbilisi Math. J. 10(4), 129–141 (2017).

4. WW. Breckner, Stetigkeitsaussagenf ureine Klass ever all gemeinerter konvexer funktionen in topologisc henlianeren Raumen, Pupl. nst. Math., 23 (1978), 13-20.
5. Z. Dahmani, On Minkowski and Hermite-Hadamard integral inequalities via fractional integration. Ann. Funct. Anal. 1(1), 51–58 (2010).
6. S.S. Dragomir and S. Fitzpatrick, The Hadamards inequality for s-convex functions in the second sense. Demonstr. Math. 32(4), 687-696 (1999).
7. H. Hudzik and L. Maligranda, Some remarks on s-convex functions, Aequationes Math., 48 (1994), 100-111.
8. C.E.M. Pearce, P-functions, Quasi-convex Functions and Hadamard-type Inequalities.Journal of Mathematical Analysis and Applications, 240(1999), 92-104.
9. S.G. Samko, et al., Fractional Integral and Derivatives, Theory and Applications, Gordon and Breach, Yverdon et alibi (1993).
10. M.Z. Sarikaya, E. Set, H. Yaldiz, N. Basak, Hermiteâ€“Hadamardâ€™s inequalities for fractional integrals and related fractional inequalities, Mathematical and Computer Modelling, 57 (9-10), 2403-2407, (2013).
11. E. Set, A.O. Akdemir, M.E. Özdemir, Simpson type integral inequalities for convex functions via Riemann-Liouville integrals. Filomat 31(14), 4415–4420 (2017).
12. E. Set, New inequalities of Ostrowski type for mappings whose derivatives are s-convex in the second sense via fractional integrals, Computers and Mathematics with Applications, 63 (7), 1147-1154, (2012).
13. A.O. Akdemir, A. Karaoğlan, M.A. Ragusa and E. Set, Fractional integral inequalities via Atangana-Baleanu operators for convex and concave functions, Journal of Function Spaces, volume 2021, (2021).
14. E. Set, A.O. Akdemir, A. Karaoğlan, T. Abdeljawad, and W. Shatanawi, On new generalizations of Hermite-Hadamard type inequalities via Atangana-Baleanu fractional integral operators, Axioms, MDPI, 10, 223, (2021).
15. E. Set, A. O. Akdemir, M.E. Özdemir, A. Karaoğlan and M.A. Dokuyucu, New integral inequalities for Atangana-Baleanu fractional integral operators and various comparisons via simulations, Filomat, 37(7) (2023), 2251-2267.
16. H. M. Srivastava, J. Choi, Zeta and q-Zeta Functions and Associated Series and Integrals, Elsevier (2012).

Some Applications on the Darboux Vector of Viviani's Curve

Süleyman ŞENYURT[1] and Davut CANLI[1]

[1]*Department of Mathematics, Faculty of Science and Arts, Ordu University, Ordu-TÜRKİYE*

senyurtsuleyman52@gmail.com
davutcanli46@gmail.com

ABSTRACT

The study examines special Viviani's curve and its Darboux vector to construct corresponding Sabban frame for the curve. Then, as an application, new Smarandache curves are derived with respect to these Sabban vectors. The geodesic curvature for each curve is expressed by the curvature and the torsion of Viviani's curve. Graphical illustrations for each curve and their geodesic curvatures are presented to support their relationships.

1. INTRODUCTION

In differential geometry, it is often interest to derive new curves and discuss their characteristics. Smarandache geometry is a useful method of deriving new curves from a given specific curve. It exploits the orthonormal vectors moving along the given curve according to a specific frame. Thus, a curve whose position vector is a linear combination of for example Frenet vectors is called a Smarandache curve [3, 20]. By using different frames apart from Frenet, various curves are constructed by using this method, and their properties are well studied in many research papers (see e.g. [2-5, 18-19]). Sabban frame, as defined in [8], is one of these frames that can be used to generate new curves. By its definition which depends on a specific spherical indicatrix curve, different types of Sabban frames can be formed. By considering these forms of Sabban frame, new Smarandache curves are defined and their geodesic curvatures are examined in [14-15]. The Smarandache curves of some special curves such as Mannheim, Salkowski, involute-evolute, etc. are also discussed with respect to different frames in [6, 9-13]. More recently, the Smarandache curves that is obtained by Sabban frame according to the tangent indicatrix of the helix curve has been examined in [16], while Viviani's curve is discussed in [17]. Furthermore, by considering the principal normal vector of the special Viviani's curve, new Smarandache curves are introduced in [achakara]. Motivated by these papers, in this study, a new form of Sabban frame is constructed by using the unit Darboux vector of Viviani's curve as a spherical indicatrix curve. Then, according to this frame, new Smarandache curves are obtained. Before starting, we recall some basic notations:

Let $\alpha : I \subset R \to R^3$ be a differentiable curve and denote the set of its Frenet frame as $\{T, N, B\}$. The calculation of the very famous Frenet formulas is given as following

$$T = \frac{\alpha'}{\|\alpha'\|}, \qquad N = B \wedge T, \qquad B = \frac{\alpha' \wedge \alpha''}{\|\alpha' \wedge \alpha''\|}, \tag{1.1}$$

$$\kappa = \frac{\|\alpha' \wedge \alpha''\|}{\|\alpha'\|^3}, \qquad \tau = \frac{\det(\alpha', \alpha'', \alpha''')}{\|\alpha' \wedge \alpha''\|^2}, \tag{1.2}$$

$$T'(s) = \eta\kappa N, \quad N'(s) = \eta(-\kappa T + \tau B), \quad B'(s) = -\eta\tau N, \tag{1.3}$$

where $'$ is for the derivative operator $\wedge$ is vector product sign, $\eta = \|\alpha'(s)\|$, κ is the curvature function, and τ is the torsion of the curve. The motion of the Frenet frame along the curve $\alpha = \alpha(s)$ follows an instantaneous rotation vector which is called Darboux vector defined by Gaston Darboux as follows [1, 7]

$$W = \tau T + \kappa B.$$

If $\angle(B, W) = \varphi,$ then the unit Darboux vector denoted by C can be given as

$$C = \sin(\varphi)T + \cos(\varphi)B, \tag{1.4}$$

where $\cos(\varphi) = \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}}$, $\sin(\varphi) = \frac{\tau}{\sqrt{\kappa^2 + \tau^2}}$, and $\varphi' = \left(\frac{\tau}{\kappa}\right)'\left(1 + \frac{\tau^2}{\kappa^2}\right)$ [7].

Frenet frame is not the only frame to characterize the curves. There are other frames, one of which is known to be Sabban frame. The frame is defined as follows:

Let $\lambda(s): R \to R^3$ be a unit vector with its arclength parameter s. Denote $\mathrm{I} = \lambda'$ as the tangent vector of the curve whose position vector $\lambda(s)$ on S^2. By using vector product, another unit vector is computed as $\prod = \lambda \wedge \mathrm{I}$. These three vectors construct an orthonormal frame denoted by $\{\lambda, \mathrm{I}, \prod\}$. This new frame is known as Sabban frame [8]. The following relations are Frenet like formulas for Sabban frame

$$\lambda' = \mathrm{I}, \qquad \mathrm{I}' = -\lambda + \aleph_g \prod, \qquad \prod{}' = -\aleph_g \mathrm{I}, \tag{1.5}$$

where

$$\aleph_g = \langle \mathrm{I}', \prod \rangle \text{ [8, 19].} \tag{1.6}$$

In addition, the locus of the intersection points of a unit sphere centered at origin with a given cylinder tangent to this sphere form a special curve namely Viviani's curve (see Figure 1).

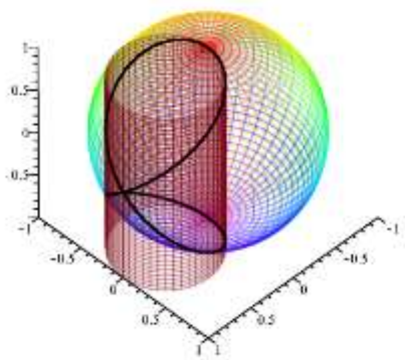


Figure 1: The special Viviani's curve as an intersection of sphere and cylinder

The parametric representation of the curve is as follows

$$\alpha(s)=\left(\cos^2 s,\cos s\sin s,\sin s\right),\quad s\in[-\pi,\pi]\ [1].$$

The Frenet apparatus and the corresponding unit Darboux vector for Viviani's curve is given as follows:

$$T(s)=\frac{2\left(-\sin 2s,\cos 2s,\cos s\right)}{\sqrt{2\cos 2s+6}},\quad N(s)=-\frac{\left(\cos 4s+12\cos 2s+3,\sin 4s+12\sin 2s,4\sin s\right)}{\sqrt{6\cos 4s+88\cos 2s+162}},$$

$$B(s)=\frac{\left(\sin 3s+3\sin s,-\cos 3s-3\cos s,4\right)}{\sqrt{6\cos 2s+26}},$$

$$\kappa(s)=\frac{\sqrt{3\cos^2 s+5}}{\left(\cos^2 s+1\right)\sqrt{\cos^2 s+1}},\quad \tau(s)=\frac{6\cos s}{3\cos^2 s+5},$$

$$C(s)=\frac{\left(2\sqrt{2}\sin(s)^3\left(6\cos(s)^2+5\right),4\sqrt{2}\cos(s)\left(3\cos(s)^4-2\cos(s)^2-3\right),\sqrt{2}\left(3\cos(2s)^2+18\cos(2s)+35\right)\right)}{\sqrt{18\cos(2s)^4+207\cos(2s)^3+999\cos(2s)^2+2493\cos(2s)+2683}}.$$

2. ON SABBAN FRAME OF VIVIANI'S CURVE BY UNIT DARBOUX VECTOR AND SOME APPLICATIONS

This is the main section of the paper where we first construct Sabban frame by considering the pol-indicatrix curve drawn by the unit Darboux vector of the Viviani's curve. Then, we provide the relationship among the Frenet and Sabban vectors. Second, by using the vectors of Sabban frame as position vectors, some special Smarandache curves are defined. Finally, the geodesic curvatures of new curves are expressed by the curvature and the torsion of Viviani's curve.

Definition 1. *Let $\alpha: s\in I\subset R\to R^3$ be a space curve twice differentiable everywhere in its domain and denote its unit Darboux vector by $C(s)$. The curve traced out by the vector C of α centered at unit sphere is called pol- or Darboux- indicatrix curve $\alpha_C(s)=C(s)$ (see Figure 2).*

Theorem 1. *Let $\alpha_C(s)=C(s)$ be the pol-indicatrix curve of α and denote $\{T_C,N_C,B_C\}$ as the Frenet frame of it. Then, the relationship among the actual Frenet frame of α and its pol-indicatrix curve can be given as follows*

$$\begin{aligned}
T_C &= \cos\varphi T-\sin\varphi B,\\
N_C &= \frac{\varphi'\left(\sin\varphi T+\cos\varphi B\right)+\eta\sqrt{\kappa^2+\tau^2}N}{\sqrt{2\eta^2\left(\kappa^2+\tau^2\right)+\varphi'^2}},\\
B_C &= \frac{\eta\sqrt{\kappa^2+\tau^2}\left(\sin\varphi T+\cos\varphi B\right)+\varphi' N}{\sqrt{2\eta^2\left(\kappa^2+\tau^2\right)+\varphi'^2}}.
\end{aligned}\tag{2.1}$$

Proof. From Definition 1. of pol-indicatrix curve as $\alpha_C(s)=C(s)$, by taking derivatives with consideration to the relations (1.3), we compute the followings

$$
\begin{aligned}
&\alpha_C' = \varphi'(\cos\varphi T - \sin\varphi B),\\
&\alpha_C'' = \left(\varphi'\cos\varphi - \varphi'^2\sin\varphi\right)T + \varphi'\eta\sqrt{\kappa^2+\tau^2}N - \left(\varphi'\sin\varphi + \varphi'^2\cos\varphi\right)B,\\
&\alpha_C' \wedge \alpha_C'' = \varphi'^2\eta\sqrt{\kappa^2+\tau^2}\sin\varphi T + \varphi'^2\eta\sqrt{\kappa^2+\tau^2}\cos\varphi B + \varphi'^3 N,\\
&\left\|\alpha_C' \wedge \alpha_C''\right\| = \varphi'^2\sqrt{2\eta^2\left(\kappa^2+\tau^2\right)+\varphi'^2}.
\end{aligned}
\tag{2.2}
$$

By referring the relations (1.1) to substitute these relations, the proof is completed.

Theorem 2. *The curvatures of the pol-indicatrix curve* $\alpha_C(s) = C(s)$ *denoted by* κ_C *and* τ_C, *respectively can be expressed by the curvatures of* α *as follows:*

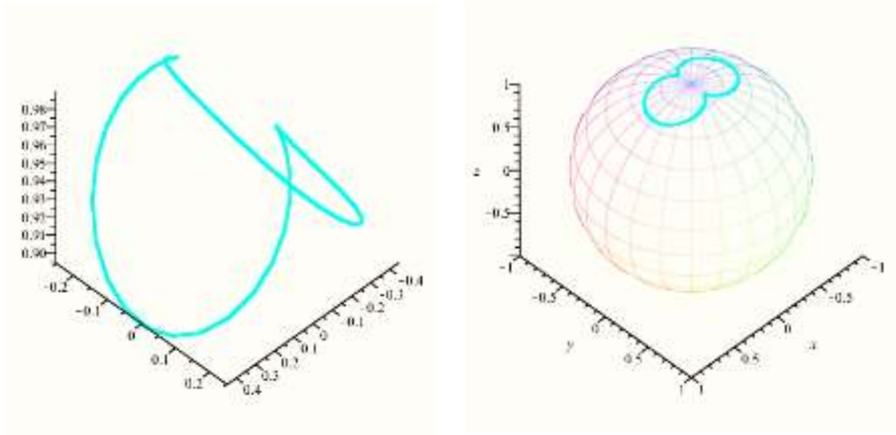
$$
\kappa_C = \frac{\sqrt{2\eta^2\left(\kappa^2+\tau^2\right)+\varphi'^2}}{\varphi'}, \qquad \tau_C = \frac{\eta'\sqrt{\kappa^2+\tau^2} - \eta\left(\sqrt{\kappa^2+\tau^2}\right)'}{2\eta^2\left(\kappa^2+\tau^2\right)+\varphi'^2}.
\tag{2.3}
$$

Proof. By taking the third derivative of the pol-indicatrix curve and using triple product, we have

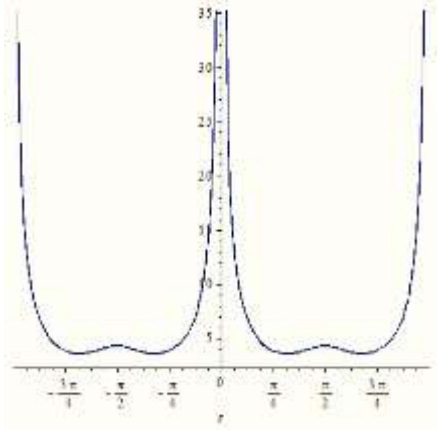
$$
\begin{aligned}
\alpha_C''' = &\left(\left(\varphi'\cos\varphi - \varphi'^2\sin\varphi\right)' - \varphi'\kappa\eta^2\sqrt{\kappa^2+\tau^2}\right)T + \left(\varphi'\eta\sqrt{\kappa^2+\tau^2} + \left(\varphi'\eta\sqrt{\kappa^2+\tau^2}\right)'\right)N\\
&-\left(\left(\varphi'\sin\varphi + \varphi'^2\cos\varphi\right)' - \varphi'\tau\eta^2\sqrt{\kappa^2+\tau^2}\right)B,\\
\det\left(\alpha_C', \alpha_C'', \alpha_C'''\right) = &\ \varphi'^3\left(\left(\varphi'\eta\sqrt{\kappa^2+\tau^2}\right)' - \varphi''\eta\sqrt{\kappa^2+\tau^2}\right).
\end{aligned}
$$

As we refer the relations given in (1.2) and (2.2), and by using the latter ones above, the proof is completed.

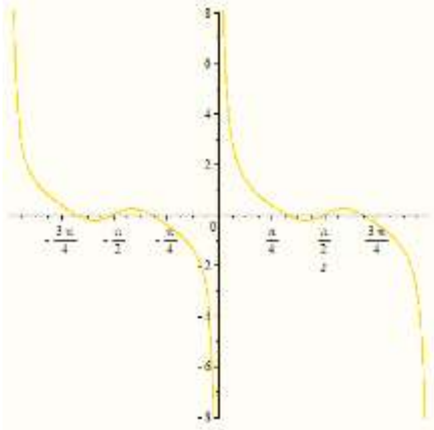
The following Figure 2 illustrates the pol-indicatrix curve, its view on unit sphere and its curvatures, respectively.



$\alpha_C(s) = C(s)$ $\qquad$ $\alpha_C(s) = C(s)$ on S^2



κ_C



τ_C

Figure 2. The pol-indicatrix curve and its curvatures

Now, let T_C denote the tangent vector of pol-indicatrix curve $\alpha_C(s)=C(s)$ and compute $D_C=\alpha_C\wedge T_C$. Then the corresponding Sabban vectors can be obtained as follows:

$$\alpha_C = sin\varphi T+\cos\varphi B,\quad T_C=\cos\varphi T - sin\varphi B,\quad D_C=N. \tag{2.4}$$

Furthermore, by using (1.5), Sabban derivative formulas can be given by

$$\begin{aligned}\alpha_c' &= \varphi'(\cos\varphi T-\sin\varphi B),\\ T_C' &= -\varphi'(\sin\varphi T+\cos\varphi B)+\eta\sqrt{\kappa^2+\tau^2}N,\\ D_C' &= \eta(-\kappa T+\tau N).\end{aligned} \tag{2.5}$$

Definition 2. *The curve whose position vector is the unit combination of* α_C *and* T_C *is called* σ_1 – *Smarandache curve which is given by the following parameterization*

$$\sigma_1(s)=\frac{1}{\sqrt{2}}(\alpha_C+T_C). \tag{2.6}$$

Note that, from (2.4), we can re-express the σ_1 – Smarandache curve as

$$\sigma_1(s)=\frac{(\cos\varphi+\sin\varphi)T+(\cos\varphi-\sin\varphi)B}{\sqrt{2}}.$$

Theorem 2. *The geodesic curvature* $K_g^{\sigma_1(s)}$ *of the* σ_1 – *Smarandache curve is given by the following relation*

$$K_g^{\sigma_1}=\frac{2\varphi'\eta\left(\sqrt{\kappa^2+\tau^2}\right)'+\left(\eta^3\left(\kappa^2+\tau^2\right)+2\varphi'\eta'+2\varphi'^2\eta-2\eta\varphi''\right)\sqrt{\kappa^2+\tau^2}}{\sqrt{2}\left(\eta^2\left(\kappa^2+\tau^2\right)-2\varphi'^2\sin 2\varphi\right)}.$$

Proof. From Definition 2, by taking the derivative of (2.6) and considering the given relations (2.5), the tangent of the σ_1 – Smarandache curve denoted by T_{σ_1} can be obtained as

$$T_{\sigma_1}=\frac{\varphi'(\cos\varphi-\sin\varphi)T+\eta\sqrt{\kappa^2+\tau^2}N-\varphi'(\cos\varphi+\sin\varphi)B}{\sqrt{\eta^2(\kappa^2+\tau^2)-2\varphi'^2\sin2\varphi}}$$
$$=\frac{\varphi'(\cos\varphi-\sin\varphi)T+\eta\sqrt{\kappa^2+\tau^2}N-\varphi'(\cos\varphi+\sin\varphi)B}{X},$$

where $X=\sqrt{\eta^2(\kappa^2+\tau^2)-2\varphi'^2\sin2\varphi}$. By using the cross product of σ_1 and T_{σ_1}, we have

$$\sigma_1\wedge T_{\sigma_1}=\frac{\eta\sqrt{\kappa^2+\tau^2}(\sin\varphi-\cos\varphi)T+2\varphi'N+\eta\sqrt{\kappa^2+\tau^2}(\sin\varphi+\cos\varphi)B}{\sqrt{2}X}.$$

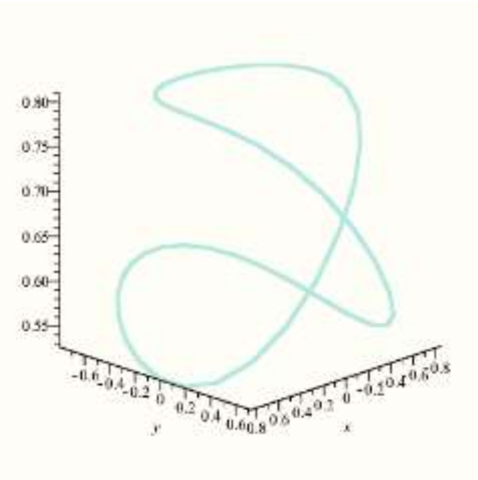
Moreover, the derivative of T_{σ_1} is

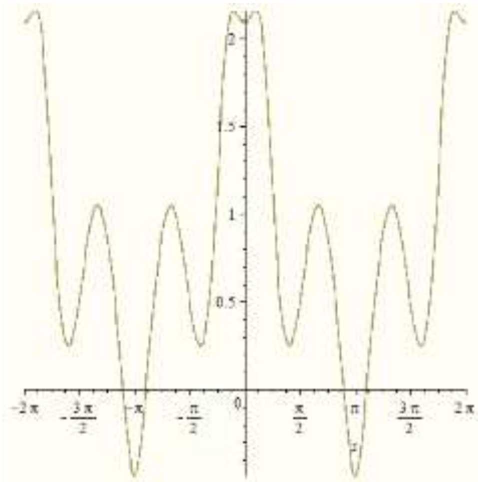
$$T'_{\sigma_1}=\left(\left(\frac{\varphi'(\cos\varphi-\sin\varphi)}{X}\right)'-\frac{\eta^2\kappa\sqrt{\kappa^2+\tau^2}}{X}\right)T+\left(\left(\frac{\eta\sqrt{\kappa^2+\tau^2}}{X}\right)'+\frac{\varphi'\eta\sqrt{\kappa^2+\tau^2}}{X}\right)N-\left(\left(\frac{\varphi'(\cos\varphi+\sin\varphi)}{X}\right)'-\frac{\eta^2\tau\sqrt{\kappa^2+\tau^2}}{X}\right)B$$
$$=\left(\frac{(\varphi''X-\varphi'X')(\cos\varphi-\sin\varphi)-\varphi'^2X(\cos\varphi+\sin\varphi)-\eta^2\kappa X\sqrt{\kappa^2+\tau^2}}{X^2}\right)T$$
$$+\left(\frac{(\eta'X-\eta X')\sqrt{\kappa^2+\tau^2}+\eta X\left(\sqrt{\kappa^2+\tau^2}\right)'+X\varphi'\eta\sqrt{\kappa^2+\tau^2}}{X^2}\right)N$$
$$-\left(\frac{(\varphi''X-\varphi'X')(\cos\varphi+\sin\varphi)+\varphi'^2X(\cos\varphi-\sin\varphi)-X\eta^2\tau\sqrt{\kappa^2+\tau^2}}{X^2}\right)B$$

Finally, by recalling (1.6), we have

$$K_g^{\sigma_1}=\left(\frac{\eta\sqrt{\kappa^2+\tau^2}(\varphi''X-\varphi'X')(\sin\varphi-1)+\varphi'^2\eta X\sqrt{\kappa^2+\tau^2}\cos2\varphi}{\sqrt{2}X^3}-\frac{\eta^3\kappa(\kappa^2+\tau^2)(\sin\varphi-\cos\varphi)}{\sqrt{2}X^2}\right)$$
$$+\left(\frac{-\eta\sqrt{\kappa^2+\tau^2}(\varphi''X-\varphi'X')(\sin\varphi+1)-\varphi'^2\eta X\sqrt{\kappa^2+\tau^2}\cos2\varphi}{\sqrt{2}X^3}+\frac{\eta^3\tau(\kappa^2+\tau^2)(\sin\varphi+\cos\varphi)}{\sqrt{2}X^2}\right)$$
$$+\left(\frac{2\varphi'(\eta'X-\eta X')\sqrt{\kappa^2+\tau^2}+2\varphi'\eta X\left(\sqrt{\kappa^2+\tau^2}\right)'}{\sqrt{2}X^3}+\frac{2\varphi'^2\eta\sqrt{\kappa^2+\tau^2}}{\sqrt{2}X^2}\right)$$
$$=\frac{\left(\eta^3(\kappa^2+\tau^2)+2\varphi'\eta'+2\varphi'^2\eta-2\eta\varphi''\right)\sqrt{\kappa^2+\tau^2}+2\varphi'\eta\left(\sqrt{\kappa^2+\tau^2}\right)'}{\sqrt{2}X^2}.$$

Upon substitution X into the latter, we complete the proof.





$\sigma_1 -$ $\qquad$ $K_g^{\sigma_1}$

Figure 3. The σ_1 – Smarandache curve (left) and its geodesic curvature (right) for $s \in [-2\pi, 2\pi]$

Definition 3. *The curve whose position vector is the unit combination of* α_C *and* D_C *is called* σ_2 *– Smarandache curve which is given by the following parameterization*

$$\sigma_2(s) = \frac{1}{\sqrt{2}}(\alpha_C + D_C). \qquad (2.7)$$

Note that, from (2.4), we can re-express the σ_2 – Smarandache curve as

$$\sigma_2(s) = \frac{\sin\varphi T + N + \cos\varphi B}{\sqrt{2}}.$$

Theorem 3. *The geodesic curvature* $K_g^{\sigma_2(s)}$ *of the* σ_2 *– Smarandache curve is given by the following relation*

$$K_g^{\sigma_2} = \frac{(\eta - \eta\varphi'^2)(\kappa^2+\tau^2) + \varphi'^3\sqrt{\kappa^2+\tau^2} + \left(\eta^2\sqrt{\kappa^2+\tau^2} - \eta\varphi'\right)(\tau'\kappa - \tau\kappa')}{\sqrt{2}\sqrt{\kappa^2+\tau^2}}.$$

Proof. From Definition 3, by taking the derivative of (2.7) and considering the given relations (2.5), the tangent of the σ_2 – Smarandache curve denoted by T_{σ_2} can be obtained as

$$T_{\sigma_2} = \frac{(\varphi'\cos\varphi - \eta\kappa)T + (-\varphi'\sin\varphi + \eta\tau)B}{\sqrt{\varphi'^2 + \eta^2(\kappa^2+\tau^2) - 2\eta\varphi'\sqrt{\kappa^2+\tau^2}}}.$$

By using the cross product of σ_2 and T_{σ_2}, we have

$$\sigma_2 \wedge T_{\sigma_2} = \frac{(\eta\tau - \varphi'\sin\varphi)T + \left(\varphi' - \eta\sqrt{\kappa^2+\tau^2}\right)N + (\eta\kappa - \varphi'\cos\varphi)B}{\sqrt{2}\sqrt{\varphi'^2 + \eta^2(\kappa^2+\tau^2) - 2\eta\varphi'\sqrt{\kappa^2+\tau^2}}}.$$

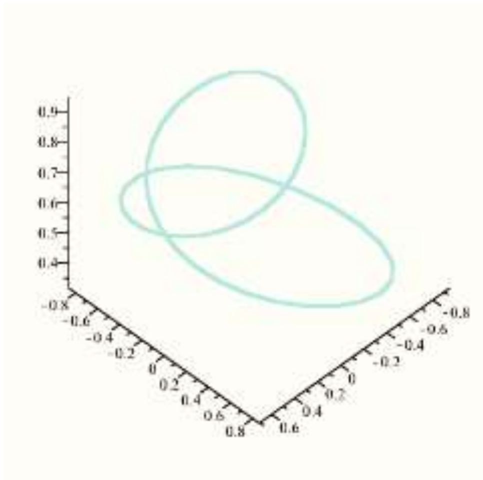
Besides, the derivative of T_{σ_2} is

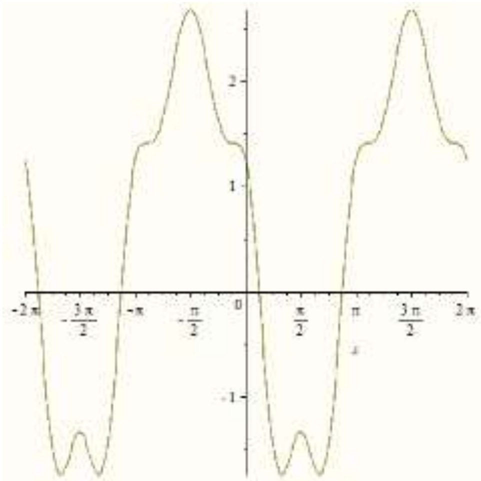
$$T'_{\sigma_2} = \left(\frac{\varphi' \cos\varphi - \eta\kappa}{\sqrt{\varphi'^2 + \eta^2\left(\kappa^2+\tau^2\right) - 2\eta\varphi'\sqrt{\kappa^2+\tau^2}}} \right)' T + \left(\frac{-\varphi' \sin\varphi + \eta\tau}{\sqrt{\varphi'^2 + \eta^2\left(\kappa^2+\tau^2\right) - 2\eta\varphi'\sqrt{\kappa^2+\tau^2}}} \right)' B$$
$$+ \left(\frac{\varphi'\eta\sqrt{\kappa^2+\tau^2} - \eta^2\left(\kappa^2+\tau^2\right)}{\sqrt{\varphi'^2 + \eta^2\left(\kappa^2+\tau^2\right) - 2\eta\varphi'\sqrt{\kappa^2+\tau^2}}} \right) N.$$

By using (1.6), we have

$$K_g^{\sigma_2} = \frac{\left(\eta - \eta\varphi'^2\right)\left(\kappa^2+\tau^2\right) + \varphi'^3\sqrt{\kappa^2+\tau^2} + \left(\eta^2\sqrt{\kappa^2+\tau^2} - \eta\varphi'\right)\left(\tau'\kappa - \tau\kappa'\right)}{\sqrt{2}\sqrt{\kappa^2+\tau^2}},$$

which completes the proof.





$\sigma_2 -$ $\qquad\qquad$ $K_g^{\sigma_2}$

Figure 4. The σ_2 – Smarandache curve (left) and its geodesic curvature (right) for $s \in [-2\pi, 2\pi]$

Definition 4. *The curve whose position vector is the unit combination of* T_C *and* D_C *is called* σ_3 *– Smarandache curve which is given by the following parameterization*

$$\sigma_3(s) = \frac{1}{\sqrt{2}}\left(T_C + D_C\right). \tag{2.8}$$

Note that, from (2.4), we can re-express the σ_3 – Smarandache curve as

$$\sigma_3(s) = \frac{\cos\varphi T + N - \sin\varphi B}{\sqrt{2}}.$$

Theorem 4. *The geodesic curvature* $K_g^{\sigma_3(s)}$ *of the* σ_3 *– Smarandache curve is given by the following relation*

$$K_g^{\sigma_3} = \frac{\varphi'^3 + 2\eta^2\left(\tau'\kappa - \tau\kappa\right) + 2\left(\varphi'\eta' - \varphi''\eta\right)\sqrt{\kappa^2+\tau^2} + 2\varphi'\eta\sqrt{\kappa^2+\tau^2}'}{\sqrt{2}\left(\varphi'^2 + \eta^2\left(\kappa^2+\tau^2\right)\right)}.$$

Proof. By taking the derivative of (2.8) from Definition 4, and by considering the given relations (2.5), the tangent of the σ_3 – Smarandache curve denoted by T_{σ_3} can be obtained as

$$T_{\sigma_3} = \frac{\left(-\varphi'\sin\varphi - \eta\kappa\right)T + \eta\sqrt{\kappa^2+\tau^2}N + \left(-\varphi'\cos\varphi + \eta\tau\right)B}{Y},$$

where $Y = \sqrt{\varphi'^2 + \eta^2\left(\kappa^2+\tau^2\right)}$. By using the cross product of σ_3 and T_{σ_3}, we have

$$\sigma_3 \wedge T_{\sigma_3} = \frac{\left(-\varphi'\cos\varphi + \eta\sqrt{\kappa^2+\tau^2}\sin\varphi + \eta\tau\right)T + \varphi'N + \left(\varphi'\sin\varphi + \eta\sqrt{\kappa^2+\tau^2}\cos\varphi + \eta\kappa\right)B}{\sqrt{2}Y}.$$

Furthermore, the derivative of T_{σ_3} is

$$\begin{aligned}
T'_{\sigma_3} &= -\left(\frac{\eta^2\kappa\sqrt{\kappa^2+\tau^2}}{Y} + \left(\frac{\varphi'\sin\varphi + \eta\kappa}{Y}\right)'\right)T + \left(-\frac{\eta^2\left(\kappa^2+\tau^2\right)}{Y} + \left(\frac{\eta\sqrt{\kappa^2+\tau^2}}{Y}\right)'\right)N + \left(\frac{\eta^2\tau\sqrt{\kappa^2+\tau^2}}{Y} + \left(\frac{-\varphi'\cos\varphi + \eta\tau}{Y}\right)'\right)B \\
&= -\left(\frac{\eta^2\kappa\sqrt{\kappa^2+\tau^2}}{Y} + \frac{Y\left(\varphi''\sin\varphi + \varphi'^2\cos\varphi + (\eta\kappa)'\right) - Y'\left(\varphi'\sin\varphi + \eta\kappa\right)}{Y^2}\right)T \\
&\quad + \left(-\frac{\eta^2\left(\kappa^2+\tau^2\right)}{Y} + \frac{Y\left(\eta'\sqrt{\kappa^2+\tau^2} + \eta\sqrt{\kappa^2+\tau^2}'\right) - Y'\eta\sqrt{\kappa^2+\tau^2}}{Y^2}\right)N \\
&\quad + \left(\frac{\eta^2\tau\sqrt{\kappa^2+\tau^2}}{Y} + \frac{Y\left(-\varphi''\cos\varphi + \varphi'^2\sin\varphi + (\eta\tau)'\right) - Y'\left(-\varphi'\cos\varphi + \eta\tau\right)}{Y^2}\right)B.
\end{aligned}$$

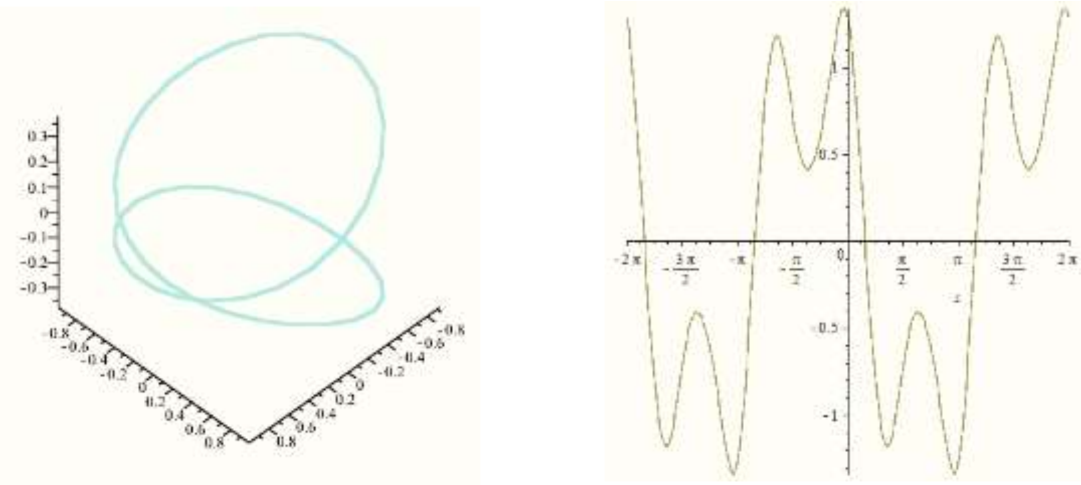
Last, from (1.6), we have

$$K_g^{\sigma_3} = \frac{\begin{array}{c}\left(-\varphi''\cos\varphi + \varphi'^2\sin\varphi + (\eta\tau)'\right)\left(\varphi'\sin\varphi + \eta\sqrt{\kappa^2+\tau^2}\cos\varphi + \eta\kappa\right) \\ -\left(\varphi''\sin\varphi + \varphi'^2\cos\varphi + (\eta\kappa)'\right)\left(-\varphi'\cos\varphi + \eta\sqrt{\kappa^2+\tau^2}\sin\varphi + \eta\tau\right) + \varphi'\left(\eta'\sqrt{\kappa^2+\tau^2} + \eta\sqrt{\kappa^2+\tau^2}'\right)\end{array}}{\sqrt{2}Y^2}$$

,
and therefore

$$K_g^{\sigma_3} = \frac{\varphi'^3 + 2\eta^2\left(\tau'\kappa - \tau\kappa\right) + 2\left(\varphi'\eta' - \varphi''\eta\right)\sqrt{\kappa^2+\tau^2} + 2\varphi'\eta\sqrt{\kappa^2+\tau^2}'}{\sqrt{2}Y^2}.$$

Upon substitution of Y into above relation we complete the proof.



$\sigma_3 -$ $\qquad\qquad$ $K_g^{\sigma_3}$

Figure 5. The σ_3 – Smarandache curve (left) and its geodesic curvature (right) for $s \in [-2\pi, 2\pi]$

Definition 5. *The curve whose position vector is the unit combination of* α_C, T_C *and* D_C *is called* σ_4 *– Smarandache curve which is given by the following parameterization*

$$\sigma_4(s) = \frac{1}{\sqrt{3}}(\alpha_C + T_C + D_C). \tag{2.9}$$

Note that, from (2.4), we can re-express the σ_3 – Smarandache curve as

$$\sigma_4(s) = \frac{(\cos\varphi + sin\varphi)T + N + (\cos\varphi - sin\varphi)B}{\sqrt{3}}.$$

Theorem 5. *The geodesic curvature* $K_g^{\sigma_4(s)}$ *of the* σ_4 *– Smarandache curve is given by the following relation*

$$\begin{aligned}K_g^{\delta_4} = &\left(\frac{\left(\Sigma\varphi'' - \mathrm{P}\varphi'^2 - (\eta\kappa)' - \eta^2\kappa W\right)F - (\Sigma\varphi' - \eta\kappa)F'}{2\sqrt{3}F^3}\right)(\eta\tau - \mathrm{P}\varphi' - \eta W\Sigma)\\ &+\left(\frac{\left(F\left(\eta\varphi' - \eta^2\right)W^2 + \eta' W + \eta W'\right)F - \eta W F'}{2\sqrt{3}F^3}\right)(2\varphi' - \eta W)\\ &-\left(\frac{\left(\mathrm{P}\varphi'' + \Sigma\varphi'^2 - (\eta\tau)' - \eta^2\tau W\right)F - (\mathrm{P}\varphi' - \eta\tau)F'}{2\sqrt{3}F^3}\right)(\eta\kappa - \Sigma\varphi' + \eta W \mathrm{P}),\end{aligned}$$

where $\mathrm{P} = \cos\varphi + \sin\varphi$, $\Sigma = \cos\varphi - \sin\varphi$, $W = \sqrt{\kappa^2 + \tau^2}$, $F = \sqrt{\varphi'^2 + \eta^2 W^2 - \varphi'\eta W}$.

Proof. By differentiating (2.9) of Definition 5, and by referring the relations (2.5), the tangent vector T_{σ_4} of the σ_4 – Smarandache curve is obtained

$$\begin{aligned} T_{\sigma_4} &= \frac{\left((\cos\varphi - \sin\varphi)\varphi' - \eta\kappa\right)T + \eta\sqrt{\kappa^2+\tau^2}N - \left((\cos\varphi+\sin\varphi)\varphi' - \eta\tau\right)B}{\sqrt{2}\sqrt{\varphi'^2 + \eta^2\left(\kappa^2+\tau^2\right) - \varphi'\eta\sqrt{\kappa^2+\tau^2}}}, \\ &= \frac{(\Sigma\varphi' - \eta\kappa)T + \eta WN - (\mathrm{P}\varphi' - \eta\tau)B}{\sqrt{2F}}. \end{aligned}$$

By using the cross product of σ_4 and T_{σ_4}, we have

$$\begin{aligned} \sigma_4 \wedge T_{\sigma_4} &= \frac{\begin{array}{c}\left(\eta\tau - \varphi'(\cos\varphi+\sin\varphi) + \eta(\sin\varphi - \cos\varphi)\sqrt{\kappa^2+\tau^2}\right)T + \left(2\varphi' - \eta\sqrt{\kappa^2+\tau^2}\right)N \\ +\left(\eta\kappa - \varphi'(\cos\varphi - \sin\varphi) + \eta(\cos\varphi+\sin\varphi)\sqrt{\kappa^2+\tau^2}\right)B\end{array}}{\sqrt{6F}} \\ &= \frac{(\eta\tau - \mathrm{P}\varphi' - \eta W\Sigma)T + (2\varphi' - \eta W)N + (\eta\kappa - \Sigma\varphi' + \eta W\mathrm{P})B}{\sqrt{6F}}. \end{aligned}$$

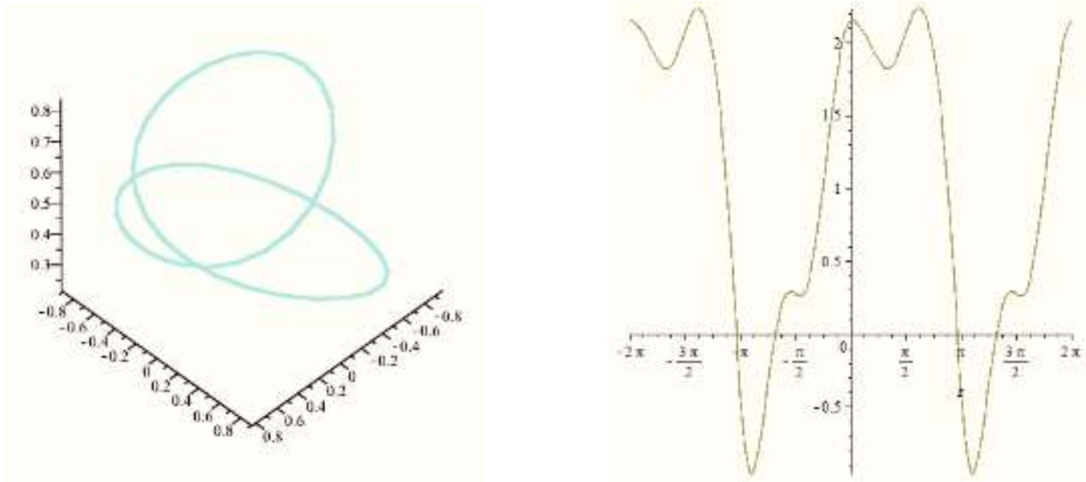
Furthermore, the derivative of T_{σ_4} is

$$\begin{aligned} T'_{\sigma_4} &= \left(\left(\frac{\Sigma\varphi' - \eta\kappa}{\sqrt{2F}}\right)' - \frac{\eta^2\kappa W}{\sqrt{2F}}\right)T + \left(\left(\frac{\eta W}{\sqrt{2F}}\right)' + \frac{(\eta\varphi' - \eta^2)W^2}{\sqrt{2F}}\right)N - \left(\left(\frac{\mathrm{P}\varphi' - \eta\tau}{\sqrt{2F}}\right)' - \frac{\eta^2\tau W}{\sqrt{2F}}\right)B \\ &= \frac{\left(\varphi''(\cos\varphi - \sin\varphi) - \varphi'^2(\cos\varphi+\sin\varphi) - (\eta\kappa)' - \eta^2\kappa\sqrt{\kappa^2+\tau^2}\right)F - \left(\varphi'(\cos\varphi - \sin\varphi) - \eta\kappa\right)F'}{\sqrt{2}F^2}T \\ &+ \frac{\left(F(\eta\varphi' - \eta^2)(\tau^2+\kappa^2) + \eta'\sqrt{\kappa^2+\tau^2} + \eta\sqrt{\kappa^2+\tau^2}'\right)F - \eta\sqrt{\kappa^2+\tau^2}F'}{\sqrt{2}F^2}N \\ &- \frac{\left(\varphi''(\cos\varphi+\sin\varphi) + \varphi'^2(\cos\varphi - \sin\varphi) - (\eta\tau)' - \eta^2\tau\sqrt{\kappa^2+\tau^2}\right)F - \left(\varphi'(\cos\varphi+\sin\varphi) - \eta\tau\right)F'}{\sqrt{2}F^2}B, \end{aligned}$$

or equivalently

$$\begin{aligned} T'_{\sigma_4} &= \frac{\left(\Sigma\varphi'' - \mathrm{P}\varphi'^2 - (\eta\kappa)' - \eta^2\kappa W\right)F - (\Sigma\varphi' - \eta\kappa)F'}{\sqrt{2}F^2}T + \frac{\left(F(\eta\varphi' - \eta^2)W^2 + \eta' W + \eta W'\right)F - \eta WF'}{\sqrt{2}F^2}N \\ &- \frac{\left(\mathrm{P}\varphi'' + \Sigma\varphi'^2 - (\eta\tau)' - \eta^2\tau W\right)F - (\mathrm{P}\varphi' - \eta\tau)F'}{\sqrt{2}F^2}B. \end{aligned}$$

Finally, from (1.6), we can obtain the expression given in theorem, which completes the proof.



$\sigma_4 -$ $\qquad\qquad\qquad\qquad$ $K_g^{\sigma_4}$

Figure 6. The σ_4 – Smarandache curve (left) and its geodesic curvature (right) for $s \in [-2\pi, 2\pi]$

REFERENCES

1. E. Abbena, S. Salamon and A. Gray Modern differential geometry of curves and surfaces with Mathematica, CRC press, 2017.
2. Ş. Alıç and B. Yılmaz, Smarandache Curves According to Alternative Frame in E^3, Journal of Universal Mathematics vol. 4, no. 2, pp. 140--156, 2021.
3. T. Ali, Special Smarandache Curves in the Euclidean Space, International Journal of Mathematical Combinatorics vol. 2, pp. 30--36, 2010.
4. Ö. Bektaş and S. Yüce, Special Smarandache Curves According to Darboux Frame in E^3, Romanian Journal of Mathematics and Computer Science vol.3, no. 1, pp. 48--59, 2013.
5. M. Çetin, Tunçer, Y. and M. K. Karacan, Smarandache curves according to Bishop frame in Euclidean 3-space, Gen. Math. Notes, vol. 20, no. 2, pp. 50--66, 2014.
6. M. Çetin and H. Kocayiğit, On the Quaternionic Smarandache Curves in Euclidean 3-Space Int. J. Contemp. Math. Sciences vol. 8, no. 3, 139--150, 2013.
7. M. P. do Carmo, Differential Geometry of Curves and Surfaces, Prentice Hall, Englewood Cliffs, NJ, 1976.
8. J. Koenderink, Solid Shape, MIT Press, Cambridge, MA, 1990.
9. S. Şenyurt and A. Çalışkan, N*C*-Smarandache Curves of Mannheim Curve Couple According to Frenet Frame, International J. Math. Combin., vol. 1, pp. 1--13, 2015.
10. S. Şenyurt and S. Sivas, An Application of Smarandache Curve, Ordu University Journal of Science and Tecnology, vol. 3, no. 1, pp. 46--60, 2013.
11. S. Şenyurt and B. Öztürk, Smarandache Curves of Salkowski Curve According Frenet Frame, Turkish Journal of Mathematics and Computer Science, vol. 10, pp. 190--201, 2018.

12. S. Şenyurt, C. Cevahir and Y. Altun, On the Smarandache Curves of Spatial Quaternionic Involute Curve, Proceedings of the National Academy of Sciences, India Section A: Physical Sciences, vol. 90, pp. 827--837, 2020.
13. S. Şenyurt and A. Çalışkan, An application according to spatial quaternionic Smarandache curve, Appl Math Sci., vol. 9, no. 5, pp. 219--228, 2015.
14. S. Şenyurt, C. Cevahir and Y. Altun, Smarandache Curves According to Sabban Frame Belonging to Mannheim Curves Pair, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math., vol. 68, no. 1, pp. 500--523, 2019.
15. S. Şenyurt, D-Smarandache Curves According to The Sabban Frame of The Spherical Indicatrix Curve, Turk. J. Math. Comput. Sci. vol.9, pp. 39--49, 2018.
16. S. Şenyurt and S. Gür Mazlum, Some Applications on the Tangent Indicatrix Curve of the Helix. 2nd International İzmir Congress on Life, Engineering, and Applied Sciences Proceedings Books, BZT Academy Pub. House, İzmir, Türkiye, pp. 89--99, 2023.
17. S. Şenyurt and D. Canlı, On the Tangent Indicatrix of Special Viviani's Curve and Its Corresponding Smarandache Curves according to Sabban Frame. 10th International "Başkent" Congress on Physical, Engineering, and Applied Sciences Proceedings Book, BZT Academy Pub. House, Türkiye, Germany, pp. 111--122, 2023.
18. S. Şenyurt, K. H. Ayvacı and D. Canlı, Smarandache Curves According to Flc-frame in Euclidean 3-space, Fundamentals of Contemporary Mathematical Sciences, vol. 4, no. 1, pp. 16--30, 2023.
19. K. Taşköprü and M. Tosun, Smarandache curves according to Sabban frame on S^2, Boletim da Sociedade Paranaense de Matematica, vol. 32, no. 1, 51--59, 2014.
20. M. Turgut and S. Yılmaz, Smarandache Curves in Minkowski Space-Time, International J. Math. Combin., vol. 3, pp. 51--55, 2008.

Novel Simpson Type Inequalities for Fractional Integrals with Respect to Another Function

Hüseyin BUDAK[1] and Mehmet Zeki SARIKAYA[1]

[1]Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, TÜRKİYE

hsyn.budak@gmail.com
sarikayamz@hotmail.com

ABSTRACT

In this study, we first prove two new identities for differentiable functions. Then by using these equalities, we obtain some Simpson type inequalities involving fractional integrals with respect to another function. For this aim, we use the functions whose derivatives in absolute value are convex and Hölder inequality.

1. INTRODUCTION

It is well known that the considerable number of inequalities have been established in the case of convex functions but the most famous is Simpson's inequality. The classical Simpson's inequality for four times continuously differentiable functions are expressed as follows:

Theorem 1. *Let us note that $f:[a,b] \to \mathbb{R}$ is a four times continuously differentiable function on (a,b), and let us consider $\left\|f^{(4)}\right\|_\infty = \sup_{x\in(a,b)} \left|f^{(4)}(x)\right| < \infty$. Then, the following inequality holds:*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{1}{2880}\left\|f^{(4)}\right\|_\infty (b-a)^4.$$

Since the convex theory is an effective and useful way to solve a large number of problems from different branches of mathematics, many mathematicians have investigated the Simpson-type inequalities the case of convex function. More precisely, some inequalities of Simpson's type for s -convex functions are established by using differentiable functions in the paper [3]. Furthermore, the new variants of Simpson's type inequalities based on differentiable convex functions are established in the papers [34, 36]. The reader is referred to [8, 14, 19, 21, 24-26, 35] and the references therein for more information and unexplained subjects about Simpson type inequalities for various convex classes.
Many mathematicians have investigated the twice differentiable convex functions for obtaining significant inequalities. For example, Sarikaya et al. proved several Simpson-type inequalities for functions whose second derivatives are convex in the paper [33]. In addition, some

Simpson's type inequalities are given for functions whose absolute values of derivatives are convex in the paper [31]. Moreover, it is proved new estimates on the generalization of Hadamard, Ostrowski, and Simpson type inequalities in the case of functions whose second derivatives in absolute value at certain powers are convex and quasi–convex functions in the paper [29]. Furthermore, Simpson type inequalities are established for P -convex functions in the paper [28]. It can be referred to [2, 11, 13, 37, 38] for further pieces of informations and unexplained subjects about these type of inequalities including twice differentiable functions.
Mathematical preliminaries about fractional calculus theory, which will be used throughout this paper, will be given as follows:

Definition 1. *Let us consider $f \in L_1[a,b]$. The Riemann–Liouville integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \ \ x > a \tag{1.1}$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \ \ x < b, \tag{1.2}$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and its described as follows:

$$\Gamma(\alpha) = \int_0^{\infty} e^{-u} u^{\alpha-1} du.$$

Let us also note that $J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x)$.

Remark 1. *If we choose $\alpha = 1$ in Definition 1, then the fractional integral reduces to the classical integral.*

Definition 2. *Let $f \in L_1[a,b]$. The Hadamard fractional integrals $\mathbf{J}_{a+}^{\alpha} f$ and $\mathbf{J}_{b-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$\mathbf{J}_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t} \right)^{\alpha-1} f(t) \frac{dt}{t}, \ x > a \tag{1.3}$$

and

$$\mathbf{J}_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(\ln \frac{t}{x} \right)^{\alpha-1} f(t) \frac{dt}{t}, \ x < b \tag{1.4}$$

respectively.

In the paper [15], the Simpson inequalities for differentiable functions are extended to Riemann-Liouville fractional integrals. In addition to these, several papers are focused on fractional

Simpson inequalities for fractional and various fractional integral operators [1, 7, 12, 17, 20, 22, 30, 32]. For further information and several properties of Riemann–Liouville fractional integrals, please refer to [4, 5, 9, 18, 23, 27].

In the paper [6], the authors proved some new inequalities of Simpson's type based on s −convexity by fractional integrals. If it is chosen $s = 1$ in the paper [6, Theorem 2.3], then it yields:

Theorem 2. *[6] Let $f : I \subset 0, \infty) \to \mathbb{R}$ denote a differentiable mapping on I^0 so that $f' \in L_1[a,b]$. Here, $a, b \in I$ with $a < b$. and if $|f'|$ is convex on $[a,b]$. Then, the following inequality for Riemann-Liouville fractional integrals holds:*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha} f\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq \frac{b-a}{2}\varphi(\alpha)\left(\left|f'(a)\right|+\left|f'(b)\right|\right).$$

Here,

$$\varphi(\alpha)=\left(\frac{2}{3}\right)^{\frac{1}{\alpha}+1}\left(\frac{\alpha}{1+\alpha}\right)+\frac{1}{2(1+\alpha)}-\frac{1}{3}.$$

Theorem 3. *[10] If we consider the assumptions of Theorem 2, then the following inequalities hold:*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{\frac{a+b}{2}+}^{\alpha} f(b)+J_{\frac{a+b}{2}-}^{\alpha} f(a)\right]\right|$$

$$\leq \frac{b-a}{12}\left[C_2(\alpha)\left(\left|f'(b)\right|+\left|f'(a)\right|\right)+2C_1(\alpha)\left|f'\left(\frac{a+b}{2}\right)\right|\right]$$

$$\leq \frac{b-a}{12}(C_1(\alpha)+C_2(\alpha))\left(\left|f'(a)\right|+\left|f'(b)\right|\right),$$

where

$$C_1(\alpha)=2\left(\frac{1}{3}\right)^{\frac{2}{\alpha}}\left[\frac{1}{2}-\frac{1}{\alpha+2}\right]+\frac{3}{\alpha+2}-\frac{1}{2},$$

$$C_2(\alpha)=2\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}\left[1-\frac{1}{\alpha+1}\right]-2\left(\frac{1}{3}\right)^{\frac{2}{\alpha}}\left[\frac{1}{2}-\frac{1}{\alpha+2}\right]+\frac{3}{(\alpha+1)(\alpha+2)}-\frac{1}{2}.$$

Iqbal et. al. [15] established new Simpson's type inequalities for Riemann–Liouville fractional integral using the convexity for the class of functions whose derivatives in absolute value at certain powers are convex functions proposed in the following theorem.

Theorem 4. [15] *Let us note that the assumptions of Theorem 2 are valid. Then, the following inequality holds:*

$$\left|\left[\frac{1}{6}f(a)+\frac{2}{3}f\left(\frac{a+b}{2}\right)+\frac{1}{6}f(b)\right]-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\right|$$

$$\leq\frac{b-a}{2^{\alpha}}(K_1+K_2)\left(\left|f'(a)\right|+\left|f'(b)\right|\right)$$

is valid. Here, $d^{\alpha}=\frac{2(2^{\alpha}-1)}{3}+1$ and

$$\begin{cases} K_1=\frac{1}{6}-\frac{1}{3}\left(1-\left(\frac{1}{3}\right)^{\frac{1}{\alpha}}\right)-\frac{1}{(\alpha+1)}\left(\frac{1}{3}\right)^{1+\frac{1}{\alpha}}+\frac{1}{2(\alpha+1)}, \\ K_2=2\left[\frac{1}{3}+\frac{1}{2(2^{\alpha}-1)}\right](d-1)+\frac{1+2^{\alpha+1}-2d^{\alpha+1}}{2(2^{\alpha}-1)(\alpha+1)}-\left(\frac{1}{3}+\frac{1}{2(2^{\alpha}-1)}\right). \end{cases}$$

The definitions of the following ψ -Hilfer fractional integrals are given in [18].

Definition 3. *Let $\psi:[a,b]\to\mathbb{R}$ be an monotone increasing function on $(a,b]$, having a continuous derivative $\psi'(x)$ on (a,b). The left-sided fractional integral of f with respect to the function ψ on $[a,b]$ of order $\alpha>0$ is defined by*

$$I_{a+;\psi}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(\psi(x)-\psi(t))^{\alpha-1}\psi'(t)f(t)dt, x>a \tag{1.5}$$

provided that the integral exists. The right-sided fractional integral of f with respect to the function ψ on $[a,b]$ of order $\alpha>0$ is defined by

$$I_{b-;\psi}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b(\psi(t)-\psi(x))^{\alpha-1}\psi'(t)f(t)dt, x<b \tag{1.6}$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function.

Remark 2. *If we choose $\psi(t)=t$, then the operators (1.5) and (1.6) reduce the Riemann-Liouville fractional operators (1.1) and (1.2), respectively.*

Remark 3. *If we choose $\psi(t) = \ln t,\ t > 0$, then the operators (1.5) and (1.6) reduce the Hadamard fractional operators (1.3) and (1.4), respectively.*

Jlelli and Samet gave the following Hermite–Hadamard inequality for ψ -Hilfer fractional integrals in [16].

Theorem 5. *Let $\psi: [a, b] \to \mathbb{R}$ be an monotone increasing function on $(a, b]$, having a continuous derivative $\psi'(x)$ on (a, b). Let f is a convex function on $[a, b]$ and $\alpha > 0$, then the following inequalities hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{4[\psi(b)-\psi(a)]^{\alpha}}\left[I^{\alpha}_{a+;\psi}F(b)+I^{\alpha}_{b-;\psi}F(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{1.7}$$

where $F(t) = f(t) + f(a + b - t)$.

The purpose of this paper is to show that some Simpson's type inequalities for ψ -Hilfer fractional integrals by convex functions. The general outline of the paper consists of three sections including the introduction. The remaining part of the paper proceeds as follows: In Section 2, we will establish two types of the Simpson's inequalities for ψ -Hilfer fractional integrals fractional integral operators by using functions whose derivatives are convex. Moreover, we also give the relations of our main findings and previous studies. In the last section, some conclusions and further directions of research are presented.

2. SIMPSON TYPE INEQUALITIES TYPE FOR Ψ -HILFER FRACTIONAL INTEGRALS

For the sake of brevity, we denote

$$M^{\alpha}_{\psi}(a,b) = \left[\psi\left(\frac{a+b}{2}\right)-\psi(a)\right]^{\alpha}$$

and

$$N^{\alpha}_{\psi}(a,b) = \left[\psi(b)-\psi\left(\frac{a+b}{2}\right)\right]^{\alpha}$$

Lemma 1. *Let $\psi: [a, b] \to \mathbb{R}$ be an monotone increasing function on $(a, b]$, having a continuous derivative $\psi'(x)$ on (a, b). Let $f: [a, b] \to R$ be a differentiable function on (a, b). Then we have the following identity*

$$\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] \tag{2.1}$$

$$-\frac{\Gamma(\alpha+1)}{2}\left[\frac{1}{M_{\psi}^{\alpha}(a,b)}I_{a+;\psi}^{\alpha}f\left(\frac{a+b}{2}\right)+\frac{1}{N_{\psi}^{\alpha}(a,b)}I_{b-;\psi}^{\alpha}f\left(\frac{a+b}{2}\right)\right]$$
$$=\frac{b-a}{2}\left[\frac{1}{N_{\psi}^{\alpha}(a,b)}\int_0^1\Xi_{\psi,\alpha}(t)f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)dt\right.$$
$$\left.-\frac{1}{M_{\psi}^{\alpha}(a,b)}\int_0^1 ô_{\psi,\alpha}(t)f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt\right],$$

where

$$\Xi_{\psi,\alpha}(t)=\frac{1}{2}\left[\psi\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)-\psi\left(\frac{a+b}{2}\right)\right]^{\alpha}-\frac{1}{3}N_{\psi}^{\alpha}(a,b)$$

and

$$ô_{\psi,\alpha}(t)=\frac{1}{2}\left[\psi\left(\frac{a+b}{2}\right)-\psi\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right]^{\alpha}-\frac{1}{3}M_{\psi}^{\alpha}(a,b).$$

Proof. By using integration by parts, we have

$$\int_0^1\Xi_{\psi,\alpha}(t)f\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)dt \tag{2.2}$$
$$=\frac{2}{b-a}\Xi_{\psi,\alpha}(t)f\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\Bigg|_0^1$$

$$-\frac{\alpha}{2}\int_0^1\left[\psi\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)-\psi\left(\frac{a+b}{2}\right)\right]^{\alpha-1}\psi'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)f\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)dt$$
$$=\frac{2}{b-a}\left[\frac{1}{6}N_{\psi}^{\alpha}(a,b)f(b)+\frac{1}{3}N_{\psi}^{\alpha}(a,b)f\left(\frac{a+b}{2}\right)\right]$$
$$-\frac{\alpha}{b-a}\int_{\frac{a+b}{2}}^{b}\left[\psi(x)-\psi\left(\frac{a+b}{2}\right)\right]^{\alpha-1}\psi'(x)f(x)dx$$
$$=\frac{2N_{\psi}^{\alpha}(a,b)}{b-a}\left[\frac{1}{6}f(b)+\frac{1}{3}f\left(\frac{a+b}{2}\right)\right]-\frac{\Gamma(\alpha+1)}{b-a}I_{b-;\psi}^{\alpha}f\left(\frac{a+b}{2}\right).$$

Similarly, we get

$$\int_0^1 \hat{o}_{\psi,\alpha}(t) f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt \tag{2.3}$$
$$=\frac{2M_\psi^\alpha(a,b)}{b-a}\left[\frac{1}{6}f(a)+\frac{1}{3}f\left(\frac{a+b}{2}\right)\right]-\frac{\Gamma(\alpha+1)}{b-a}I_{a+;\psi}^\alpha f\left(\frac{a+b}{2}\right).$$

If we multiply both sides of (2.2) and (2.3) by $\frac{b-a}{2N_\psi^\alpha(a,b)}$ and $\frac{-(b-a)}{2N_\psi^\alpha(a,b)}$, respectively, then by adding the resultant equalities, the equality (2.1) is obtained. This finishes the proof of Lemma 1.

Theorem 6. *Let consider that the assumptions of Lemma 1 hold. If the function $|f'|$ is convex on $[a,b]$, then one has the following Simpson inequality*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right.$$
$$\left.-\frac{\Gamma(\alpha+1)}{2}\left[\frac{1}{M_\psi^\alpha(a,b)}I_{a+;\psi}^\alpha f\left(\frac{a+b}{2}\right)+\frac{1}{N_\psi^\alpha(a,b)}I_{b-;\psi}^\alpha f\left(\frac{a+b}{2}\right)\right]\right|$$
$$\leq\frac{b-a}{4}\left[\frac{A_1(\psi,\alpha)+A_2(\psi,\alpha)}{N_\psi^\alpha(a,b)}\left|f'(b)\right|+\frac{A_1(\psi,\alpha)-A_2(\psi,\alpha)}{N_\psi^\alpha(a,b)}\left|f'(a)\right|\right.$$
$$\left.+\frac{B_1(\psi,\alpha)+B_2(\psi,\alpha)}{M_\psi^\alpha(a,b)}\left|f'(a)\right|+\frac{B_1(\psi,\alpha)-B_2(\psi,\alpha)}{M_\psi^\alpha(a,b)}\left|f'(b)\right|\right],$$

where

$$A_1(\psi,\alpha)=\int_0^1\left|\Xi_{\psi,\alpha}(t)\right|dt,$$

$$A_2(\psi,\alpha)=\int_0^1 t\left|\Xi_{\psi,\alpha}(t)\right|dt,$$

$$B_1(\psi,\alpha)=\int_0^1\left|\hat{o}_{\psi,\alpha}(t)\right|dt$$

and

$$B_2(\psi,\alpha)=\int_0^1 t\left|\hat{o}_{\psi,\alpha}(t)\right|dt.$$

Proof. Let us take modulus in Lemma 1. Then, we have

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right.\\&\left.-\frac{\Gamma(\alpha+1)}{2}\left[\frac{1}{M_{\psi}^{\alpha}(a,b)}I_{a+;\psi}^{\alpha}f\left(\frac{a+b}{2}\right)+\frac{1}{N_{\psi}^{\alpha}(a,b)}I_{b-;\psi}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right|\\&\leq\frac{b-a}{2}\left[\frac{1}{N_{\psi}^{\alpha}(a,b)}\int_0^1\left|\Xi_{\psi,\alpha}(t)\right|\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|dt\right.\\&\left.+\frac{1}{M_{\psi}^{\alpha}(a,b)}\int_0^1\left|\hat{O}_{\psi,\alpha}(t)\right|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt\right].\end{aligned} \tag{2.4}$$

Using the fact that $|f'|$ is convex, we obtain

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right.\\&\left.-\frac{\Gamma(\alpha+1)}{2}\left[\frac{1}{M_{\psi}^{\alpha}(a,b)}I_{a+;\psi}^{\alpha}f\left(\frac{a+b}{2}\right)+\frac{1}{N_{\psi}^{\alpha}(a,b)}I_{b-;\psi}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right|\\&\leq\frac{b-a}{2}\left[\frac{1}{N_{\psi}^{\alpha}(a,b)}\int_0^1\left|\Xi_{\psi,\alpha}(t)\right|\left[\frac{1+t}{2}\left|f'(b)\right|+\frac{1-t}{2}\left|f'(a)\right|\right]dt\right.\\&\left.+\frac{1}{M_{\psi}^{\alpha}(a,b)}\int_0^1\left|\hat{O}_{\psi,\alpha}(t)\right|\left[\frac{1+t}{2}\left|f'(a)\right|+\frac{1-t}{2}\left|f'(b)\right|\right]dt\right]\\&=\frac{b-a}{4}\left[\frac{A_1(\psi,\alpha)+A_2(\psi,\alpha)}{N_{\psi}^{\alpha}(a,b)}\left|f'(b)\right|+\frac{A_1(\psi,\alpha)-A_2(\psi,\alpha)}{N_{\psi}^{\alpha}(a,b)}\left|f'(a)\right|\right.\\&\left.+\frac{B_1(\psi,\alpha)+B_2(\psi,\alpha)}{M_{\psi}^{\alpha}(a,b)}\left|f'(a)\right|+\frac{B_1(\psi,\alpha)-B_2(\psi,\alpha)}{M_{\psi}^{\alpha}(a,b)}\left|f'(b)\right|\right].\end{aligned}$$

This finishes the proof of Theorem 6.

Remark 4. *Let us consider $\psi(t)=t$ for all $t\in[a,b]$ in Theorem 6. Then, we obtain*

$$\begin{aligned}A_1(\psi,\alpha)=B_1(\psi,\alpha)&=\left(\frac{b-a}{2}\right)^{\alpha}\int_0^1\left|\frac{t^{\alpha}}{2}-\frac{1}{3}\right|dt\\&=\left(\frac{b-a}{2}\right)^{\alpha}\left[\frac{\alpha}{\alpha+1}\left(\frac{2}{3}\right)^{1+\frac{1}{\alpha}}+\frac{1-2\alpha}{6(\alpha+1)}\right],\end{aligned}$$

$$A_2(\psi,\alpha) = B_2(\psi,\alpha) = \left(\frac{b-a}{2}\right)^{\alpha}\int_0^1 t\left|\frac{t^{\alpha}}{2}-\frac{1}{3}\right|dt$$
$$= \left(\frac{b-a}{2}\right)^{\alpha}\left[\frac{\alpha}{\alpha+2}\left(\frac{2}{3}\right)^{1+\frac{2}{\alpha}} + \frac{1-\alpha}{6(\alpha+2)}\right],$$

and

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right|$$
$$\le \frac{b-a}{2}\left[\frac{\alpha}{\alpha+1}\left(\frac{2}{3}\right)^{1+\frac{1}{\alpha}}+\frac{1-2\alpha}{6(\alpha+1)}\right]\left[\left|f'(a)\right|+\left|f'(b)\right|\right]$$

This result equivalent to Theorem 2.

Theorem 7. *Let consider that the assumptions of Lemma 1 hold. If the function* $|f'|^q$, $q>1$, *is convex on* $[a,b]$ *then one has the following Simpson inequality*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right. \tag{2.5}$$
$$\left.-\frac{\Gamma(\alpha+1)}{2}\left[\frac{1}{M_{\psi}^{\alpha}(a,b)}I_{a+;\psi}^{\alpha}f\left(\frac{a+b}{2}\right)+\frac{1}{N_{\psi}^{\alpha}(a,b)}I_{b-;\psi}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right|$$
$$\le \frac{b-a}{2N_{\psi}^{\alpha}(a,b)}\left(\int_0^1\left|\Xi_{\psi,\alpha}(t)\right|^p dt\right)^{\frac{1}{p}}\left(\frac{3\left|f'(b)\right|^q+\left|f'(a)\right|^q}{4}\right)^{\frac{1}{q}}$$
$$+\frac{b-a}{2M_{\psi}^{\alpha}(a,b)}\left(\int_0^1\left|\Xi_{\psi,\alpha}(t)\right|^p dt\right)^{\frac{1}{p}}\left(\frac{\left|f'(b)\right|^q+3\left|f'(a)\right|^q}{4}\right)^{\frac{1}{q}}$$
$$\le\left[\frac{b-a}{2N_{\psi}^{\alpha}(a,b)}\left(4\int_0^1\left|\Xi_{\psi,\alpha}(t)\right|^p dt\right)^{\frac{1}{p}}+\frac{b-a}{2M_{\psi}^{\alpha}(a,b)}\left(4\int_0^1\left|\Xi_{\psi,\alpha}(t)\right|^p dt\right)^{\frac{1}{p}}\right]\left[\left|f'(b)\right|+\left|f'(a)\right|\right]$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By well-known Hölder inequality and convexity of $|f'|^q$, we have

$$\int_0^1 \left|\Xi_{\psi,\alpha}(t)\right| \left| f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right| dt \tag{2.6}$$

$$\leq \left(\int_0^1 \left|\Xi_{\psi,\alpha}(t)\right|^p dt\right)^{\frac{1}{p}} \left(\int_0^1 \left| f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq \left(\int_0^1 \left|\Xi_{\psi,\alpha}(t)\right|^p dt\right)^{\frac{1}{p}} \left(\int_0^1 \left[\frac{1+t}{2}\left|f'(b)\right|^q+\frac{1-t}{2}\left|f'(a)\right|^q\right] dt\right)^{\frac{1}{q}}$$

$$= \left(\int_0^1 \left|\Xi_{\psi,\alpha}(t)\right|^p dt\right)^{\frac{1}{p}} \left(\frac{3\left|f'(b)\right|^q+\left|f'(a)\right|^q}{4}\right)^{\frac{1}{q}}.$$

Similarly, we get

$$\int_0^1 \left|\hat{o}_{\psi,\alpha}(t)\right| \left| f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right| dt \tag{2.7}$$

$$\leq \left(\int_0^1 \left|\hat{o}_{\psi,\alpha}(t)\right|^p dt\right)^{\frac{1}{p}} \left(\frac{\left|f'(b)\right|^q+3\left|f'(a)\right|^q}{4}\right)^{\frac{1}{q}}.$$

By substituting the inequalities (2.6) and (2.7) in (2.4), we obtain

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right.$$

$$\left.-\frac{\Gamma(\alpha+1)}{2}\left[\frac{1}{M_\psi^\alpha(a,b)} I_{a+;\psi}^\alpha f\left(\frac{a+b}{2}\right)+\frac{1}{N_\psi^\alpha(a,b)} I_{b-;\psi}^\alpha f\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq \frac{b-a}{2}\left[\frac{1}{N_\psi^\alpha(a,b)}\left(\int_0^1 \left|\Xi_{\psi,\alpha}(t)\right|^p dt\right)^{\frac{1}{p}} \left(\frac{3\left|f'(b)\right|^q+\left|f'(a)\right|^q}{4}\right)^{\frac{1}{q}}\right.$$

$$\left.+\frac{1}{M_\psi^\alpha(a,b)}\left(\int_0^1 \left|\Xi_{\psi,\alpha}(t)\right|^p dt\right)^{\frac{1}{p}} \left(\frac{\left|f'(b)\right|^q+3\left|f'(a)\right|^q}{4}\right)^{\frac{1}{q}}\right],$$

which finish the proof of first inequaty in (2.5). For the proof of second inequality, let $a_1 = \left|f'(a)\right|^q$, $b_1 = 3\left|f'(b)\right|^q$, $a_2 = 3\left|f'(a)\right|^q$ and $b_2 = \left|f'(b)\right|^q$. Using the facts that,

$$\sum_{k=1}^{n}(a_k+b_k)^s \le \sum_{k=1}^{n}a_k^s+\sum_{k=1}^{n}b_k^s, 0\le s<1 \tag{2.8}$$

and $1+3^{\frac{1}{q}} \le 4$, then the desired result can be obtained straightforwardly. This completes the proof of Theorem 7.

3. CONCLUSION

In this paper, we have delved into the concept of ψ-Hilfer fractional integrals, which serves as a generalized framework encompassing well-known fractional integrals such as Riemann-Liouville and Hadamard fractional integrals, among others. Our main objective was to establish Simpson-type inequalities for ψ-Hilfer fractional integrals, focusing on functions with convex derivatives.

Throughout the course of our study, we successfully proved two fundamental identities concerning ψ-Hilfer fractional integrals for differentiable functions. These identities played a crucial role in enabling us to derive the Simpson-type inequalities. The tools we employed in our investigation included the concept of convexity and the well-known Hölder inequality.

Our findings contribute to the growing body of knowledge surrounding fractional integrals and their properties, particularly in the context of ψ-Hilfer fractional integrals. The established Simpson-type inequalities not only deepen our understanding of this mathematical concept but also open avenues for future research in this area.

Furthermore, we emphasized the significance of connections between our main discoveries and prior studies. By doing so, we ensured that our work was firmly grounded in the existing literature and built upon previous contributions to the field.

The three sections of this paper have been dedicated to introducing the topic, establishing the Simpson-type inequalities for ψ-Hilfer fractional integrals based on functions with convex derivatives, and finally, discussing the implications of our results in relation to prior research.

In conclusion, our study has shed light on the fascinating world of ψ-Hilfer fractional integrals and their associated Simpson-type inequalities. We hope that this work will inspire further exploration and research in this area, potentially leading to new applications and insights in various scientific disciplines and real-world problem-solving.

REFERENCES

1. T. Abdeljawad, S. Rashid, Z. Hammouch, İ. İscan and Y. M. Chu, *Some new Simpson-type inequalities for generalized p-convex function on fractal sets with applications*, Advances in Difference Equations, 2020(1), 2020, 1-26.
2. M.A. Ali, H. Kara, J. Tariboon, S. Asawasamrit, H. Budak, and F. Hezenci, *Some new Simpson's-Formula-Type inequalities for twice–differentiable convex functions via generalized fractional operators*, Symmetry, 13(12), 2021, 2249.
3. M. Alomari, M. Darus, and S. S. Dragomir, *New inequalities of Simpson's type for s-convex functions with applications*, RGMIA Res. Rep. Coll., vol. 12, no. 4, 2009.
4. H. Budak, F. Hezenci, and H. Kara, *On parameterized inequalities of Ostrowski and Simpson type for convex functions via generalized fractional integrals*, Mathematical Methods in the Applied Sciences, 44(17), 2021, 12522-12536.

5. H. Budak, F. Hezenci, and H. Kara, *On generalized Ostrowski, Simpson and Trapezoidal type inequalities for co–ordinated convex functions via generalized fractional integrals*, Advances in Difference Equations, 2021, 2021, 1–32.
6. J. Chen and X. Huang, *Some new inequalities of Simpson's type for s-convex functions via fractional integrals*, Filomat, 31(15), (2017), 4989–4997.
7. S. S. Dragomir, R. P. Agarwal, P. Cerone, *On Simpson's inequality and applications*, J. Inequal. Appl. 5 (2000) 533–579.
8. T. Du, Y. Li, and Z. Yang, *A generalization of Simpson's inequality via differentiable mapping using extended* (s,m)*-convex functions*, Appl. Math. Comput., 293, 2017, 358–369.
9. R. Gorenflo, F. Mainardi, *Fractional calculus: integral and differential equations of fractional order*, Springer Verlag, Wien (1997), 223-276.
10. X. Hai and S.H. Wang *Simpson type inequalities for convex function based on the generalized fractional integrals*, Turkish Journal of Inequalities, 5(1), 2021, 1-15.
11. F. Hezenci, H. Budak, and H. Kara, *New version of Fractional Simpson type inequalities for twice differentiable functions*, Advances in Difference Equations, 2021(460), 2021.
12. S. Hussain, J. Khalid, Y. M. Chu, *Some generalized fractional integral Simpson's type inequalities with applications*, AIMS Math, 5(6), (2020), 5859-5883.
13. S. Hussain and S. Qaisar, *More results on Simpson's type inequality through convexity for twice differentiable continuous mappings*, SpringerPlus, 5(1), 2016, 1-9.
14. İ. İscan, *Hermite-Hadamard and Simpson-like type inequalities for differentiable harmonically convex functions*, Journal of Mathematics, 2014, Article ID 346305, 10 pages.
15. M. Iqbal, S. Qaisar, and S. Hussain, *On Simpson's type inequalities utilizing fractional integrals*, J. Comput. Anal. Appl, 23(6), (2017) 1137-1145.
16. M. Jleli and B. Samet, *On Hermite-Hadamard type inequalities via fractional integrals of a function with respect to another function*, J. Nonlinear Sci. Appl, 9(3), 2016, 1252-1260.
17. S. Kermausuor, *Simpson's type inequalities via the Katugampola fractional integrals for s-convex functions*, Kragujevac Journal of Mathematics, 45(5), 2021, 709-720.
18. A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204, Elsevier Sci. B.V., Amsterdam, 2006.
19. Y. Li and T. Du, *Some Simpson type integral inequalities for functions whose third derivatives are* (α,m)*-GA-convex functions*, Journal of the Egyptian Mathematical Society, 24(2), 2016, 175-180.
20. B. Z. Liu, *An inequality of Simpson type*, Proc. R. Soc. A, 461 (2005), 2155-2158.
21. W. Liu, *Some Simpson type inequalities for* h *-convex and* (α,m)*-convex functions*. J. Comput. Analy. App. 16(5), 2014, 1005-1012.
22. H. Lei, G. Hu, J. Nie and T. Du, *Generalized Simpson-type inequalities considering first derivatives through the* k *-Fractional Integrals*, IAENG International Journal of Applied Mathematics, 50(3), 2020, 1-8.
23. C. Luo and T. Du, *Generalized Simpson type inequalities involving Riemann-Liouville fractional integrals and their applications*, Filomat, 34(3), (2020), 751-760.
24. M. E. Ozdemir, A. O. Akdemir and H. Kavurmacı, *On the Simpson's inequality for convex functions on the coordinates*, Turkish Journal of Analysis and Number Theory. 2014, 2(5), 165-169.
25. J. Park, *On Simpson-like type integral inequalities for differentiable preinvex functions*,

Applied Mathematical Sciences, Vol. 7, 2013, no. 121, 6009-6021.
26. M. Matloka, *Some inequalities of Simpson type for h-convex functions via fractional integrals*, Abstract and Applied Analysis, 2015, Article ID 956850, 5 pages.
27. S. Miller, B. Ross, *An introduction to the fractional calculus and fractional differential equations*, NewYork:Wiley, 1993.
28. M.E. Ozdemir and C. Yildiz, *New inequalities for Hermite-Hadamard and Simpson type with applications*, Tamkang Journal of Mathematics, 44(2), 2013, 209–216.
29. J. Park, *On Some Integral Inequalities for Twice Differentiable Quasi–Convex and Convex Functions via Fractional Integrals*, Applied Mathematical Sciences, 9(62), 2015, 3057–3069.
30. S. Rashid, A. O. Akdemir, F. Jarad, M. A. Noor and K. I. Noor, *Simpson's type integral inequalities for κ-fractional integrals and their applications*, AIMS Mathematics, 4(4), 2019, 1087-1100.
31. M.Z. Sarikaya, N. Aktan, *On the generalization of some integral inequalities and their applications*, Mathematical and computer Modelling, 54(9-10), 2011, 2175–2182.
32. M. Z. Sar kaya H. Budak S. Erden, *On new inequalities of Simpson's type for generalized convex functions*, The Korean Journal of Mathematics, 27(2), 2019, 279-295.
33. M. Z. Sarikaya, E. Set and M. E. Özdemir, *On new inequalities of Simpson's type for functions whose second derivatives absolute values are convex*, Journal of Applied Mathematics, Statistics and Informatics, 9(1), 2013, 37-45.
34. M. Z. Sarikaya, E. Set and M. E. Özdemir, *On new inequalities of Simpson's type for convex functions*, RGMIA Res. Rep. Coll. 13(2) (2010), Article2.
35. M. Z. Sarikaya and S. Bardak, *Generalized Simpson type integral inequalities*, Konuralp Journal of Mathematics, 7(1) (2019) 186-191.
36. M. Z. Sarikaya, E. Set and M. E. Özdemir, *On new inequalities of Simpson's type for s-convex functions*, Computers & Mathematics with Applications, 60(8), 2020, 2191-2199. 1-38.
37. M. Vivas-Cortez, T. Abdeljawad, P. O. Mohammed, Y. Rangel-Oliveros, *Simpson's integral inequalities for twice differentiable convex functions*, Mathematical Problems in Engineering, 2020.
38. X. You, F. Hezenci, H. Budak, and H. Kara, *New Simpson type inequalities for twice differentiable functions via generalized fractional integrals*, AIMS Mathematics, 7(3), 2021, 3959–3971.

On New Improvements of Hermite-Hadamard Inclusions through Interval-Valued Convex Functions

Hasan KARA[1], Hüseyin BUDAK[1] and Fatih HEZENCİ[1]

[1]*Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, TÜRKİYE*

hasan64kara@gmail.com
hsyn.budak@gmail.com
fatihezenci@gmail.com

ABSTRACT

This research presents a new refinement method for Hermite-Hadamard inclusions in the context of interval-valued convex functions, utilizing the weighted Jensen inclusion approach. Furthermore, the study demonstrates that special choices can lead to extensions of the Hermite-Hadamard inclusion. A practical example is provided to illustrate the principal outcomes of this approach. The proposed method offers a more accurate refinement of Hermite-Hadamard inclusions, which can facilitate the development of new mathematical techniques for interval-valued convex functions.

1. INTRODUCTION

Over the last century, integral inequalities have attracted the interest of a good many researchers because of the importance of applied and pure mathematics. For instance, Hermite–Hadamard inequalities, based on convex mappings, have an important place in many areas of mathematics, specifically optimization theory. These inequalities, described by C. Hermite and J. Hadamard, express that if $f: I \to \mathbb{R}$ is a convex mapping on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

If ϖ is concave, both of the inequalities hold in the opposite direction. The best-known results associated with these inequalities are Midpoint and Trapezoid inequalities which are frequently used in special means and estimation errors (see [10, 16]). Afterward, many authors acquired new results related to these inequalities under various conditions of the functions. Besides, some researchers examined generalizations, refinements and counterparts, and generalizations of the inequalities (1.1).

This article consists of four sections, including the introduction. In this section, we give Hermite–Hadamard inequality for real-valued functions and Hermite–Hadamard inclusion for interval-valued functions. We also mention some related works in the literature. In Section 2, we present some fundamental information about interval-valued functions. Before we begin our main conclusions, we clarify the definitions required and the concepts needed. In Section 3, we derive some new improvements in weighted inclusions of the Hermite–Hadamard type through interval-valued convex functions. We acquire new results in the case of special choices. An example is given to illustrate these results. The correctness of the inequality obtained with this example is clearly demonstrated. In the last section, we note that the opinion and technique of this work may inspire new research in this area.

On the other side, interval analysis handled as one of the methods of solving interval uncertainty is an important material that is used in mathematical and computer models. Although this theory has a long history which may be dated back to Archimedes' calculation of the circumference of a circle, a considerable study was not published in this field until the 1950s. The first book [19] about interval analysis was published by Ramon E. Moore known as the pioneer of interval calculus in 1966. Thereafter, a great many researchers started to investigate theories and applications of interval analysis. Recently, many authors have focused on integral inequalities obtained by using interval-valued functions. For example, Sadowska [25] established Hermite–Hadamard inequality for set-valued functions which is the more general version of interval-valued mappings as follows:

Theorem 1.1 *[25] Suppose that $F: [a, b] \to \mathbb{R}_I^+$ is interval–valued convex function such that $F(t) = [\underline{F}(t), \overline{F}(t)]$. Then, we have*

$$F\left(\frac{a+b}{2}\right) \supseteq \frac{1}{b-a}(IR)\int_a^b F(x)dx \supseteq \frac{F(a)+F(b)}{2}. \tag{1.2}$$

Furthermore, well-known inequalities such as Ostrowski, Minkowski Beckenbach, and their applications were presented by considering interval-valued mappings in [5, 6, 11, 12]. More precisely, some inequalities including interval-valued Riemann-Liouville fractional integrals were derived by Budak et al. in [1]. In [17], Liu et al. gave the description of interval-valued harmonically convex mappings, and so they established some Hermite–Hadamard type inequalities including interval-valued fractional integrals. On the other hand, Budak et al. prove some weighted Fejer type inclusions in [3]. For more details about this topic, one can refer to [2, 7, 8, 13, 14, 15, 18, 20, 21, 22, 27, 28, 26]

2. PRELIMINARIES

Before we begin our principal outcomes, the following definitions and concepts need to be clarified. The notion of integral of the interval-valued mappings is mentioned. Before we can understand the definition of integrals of interval-valued functions, we need to give some concepts in the following.

A function ϕ is said to be an interval-valued function of t on $[a, b]$ if it assigns a non-empty interval to each $t \in [a, b]$.

$$\varphi(t) = [\underline{\varphi}(t), \overline{\varphi}(t)]$$

A partition of $[a, b]$ is any finite ordered subset D having the form

$$D: a = t_0 < t_1 < \ldots < t_n = b.$$

The mesh of a partition D is indicated by

$$mesh(D) = \max\{t_i - t_{i-1} : i = 1,2,\ldots,n\}.$$

We denote by $D([a,b])$ the set of all partition of $[a,b]$. Suppose that $D(\delta,[a,b])$ is the set of all $D \in D([a,b])$ such that $mesh(D) < \delta$. We take an arbitrary point ξ_i in interval $[t_{i-1}, t_i]$, $i = 1,2,\ldots,n$, and we define the sum

$$S(\varphi, D, \delta) = \sum_{i=1}^{n} \varphi(\xi_i)[t_i - t_{i-1}],$$

where $\phi: [a,b] \to \mathbb{R}_I$. The sum $S(\phi, D, \delta)$ is said to be a Riemann sum of ϕ corresponding to $D \in D(\delta,[a,b])$.

Definition 2.1. ([9],[23],[24]) $\phi: [a,b] \to \mathbb{R}_I$ *is said to be an interval Riemann integrable function (IR-integrable) on* $[a,b]$ *if there exist* $A \in P$ *and* $\delta > 0$, *for each* $\varepsilon > 0$, *such that*

$$d\big(S(\varphi, D, \delta), A\big) < \varepsilon$$

for every Riemann sum S of ϕ corresponding to each $D \in D(\delta,[a,b])$ and independent of choice of $\xi_i \in [t_{i-1}, t_i]$, $1 \le i \le n$. In this case, A is called as the IR-integral of ϕ on $[a,b]$ and is denoted by

$$A = (IR)\int_a^b \varphi(t)dt.$$

The collection of all functions that are IR-integrable on $[a,b]$ will be denote by $IR_{([a,b])}$.

The next theorem explains connection between IR-integrable and Riemann integrable (R-integrable):

Theorem 2.2. *Assume that* $\phi: [a,b] \to \mathbb{R}_I$ *is an interval-valued function such that* $\phi(t) = \left[\underline{\phi}(t), \overline{\phi}(t)\right]$. $\phi \in IR_{([a,b])}$ *if and only if* $\underline{\phi}(t), \overline{\phi}(t) \in R_{([a,b])}$ *and*

$$(IR)\int_a^b \varphi(t)dt = \left[(R)\int_a^b \underline{\varphi}(t)dt, (R)\int_a^b \overline{\varphi}(t)dt\right],$$

where $R_{([a,b])}$ denotes the all R-integrable function.
It is easy to see that if $\phi(t) \subseteq \psi(t)$ for all $t \in [a,b]$, then $(IR)\int_a^b \phi(t)dt \subseteq (IR)\int_a^b \psi(t)dt$.
The inequality obtained by Budak and Kara to establish the theorem in the section on principal

outcomes is as follows:

Theorem 2.3 (Weighted Jensen Inclusion). [4] *Let $g:[a,b]\to[a,b]$ be a function from $L^\infty[a,b]$ and $w:[a,b]\to\mathbb{R}$ be non-negative functions from $L^1[a,b]$ such that $\int_a^b w(t)dt\neq 0$. If $F:[a,b]\to R_I$ is an interval–valued convex function such that $F(t)=\left[\underline{F}(t),\overline{F}(t)\right]$, then we have,*

$$F\left(\frac{1}{\int_a^b w(t)dt}\int_a^b w(t)g(t)dt\right)\supseteq\frac{1}{\int_a^b w(t)dt}(IR)\int_a^b F(g(t))w(t)dt.$$

3. PRINCIPAL OUTCOMES

In this part, we establish some weighted Hermite–Hadamard type inclusions with the help of the interval valued convex functions.

Theorem 3.1. *Let $F:[a,b]\to\mathbb{R}_I^+$ be an interval-valued convex function, such that $F(t)=\left[\underline{F}(t),\overline{F}(t)\right]$. Then we acquire*

$$F\left(\frac{a+b}{2}\right)\supseteq A_n\supseteq\frac{1}{b-a}(IR)\int_a^b F(x)dx\supseteq B_n\supseteq\frac{F(a)+F(b)}{2}.$$

where

$$A_n=\frac{1}{2^n}\sum_{i=1}^{2^n}F\left(a+i\frac{b-a}{2^n}-\frac{b-a}{2^{n+1}}\right)$$

and

$$B_n=\frac{1}{2^{n+1}}\left[F(a)+F(b)+2\sum_{i=1}^{2^n-1}F\left(\left(1-\frac{i}{2^n}\right)F(a)+\frac{i}{2^n}F(b)\right)\right].$$

Proof. With the aid of the right side of Hermite–Hadamard inclusion (1.2), we derive

$$\frac{1}{b-a}\int_a^b F(t)dt=\frac{1}{b-a}\sum_{i=1}^{2^n}\int_{a+(i-1)\frac{b-a}{2^n}}^{a+i\frac{b-a}{2^n}}F(x)dx$$

$$\supseteq \frac{1}{b-a}\sum_{i=1}^{2^n}\left(a+i\frac{b-a}{2^n}-a-(i-1)\frac{b-a}{2^n}\right)\frac{F\left(a+i\frac{b-a}{2^n}\right)+F\left(a+(i-1)\frac{b-a}{2^n}\right)}{2}$$

$$= \frac{1}{2^{n+1}}\left[\sum_{i=1}^{2^n}F\left[\left(1-\frac{i}{2^n}\right)a+\frac{i}{2^n}b\right]+F\left[\left(1-\frac{i-1}{2^n}\right)a+\frac{i-1}{2^n}b\right]\right]$$

$$= \frac{1}{2^{n+1}}\left[F(a)+F(b)+2\sum_{i=1}^{2^n-1}F\left(\left(1-\frac{i}{2^n}\right)F(a)+\frac{i}{2^n}F(b)\right)\right]$$

$$= B_n.$$

Taking into account the interval-valued convexity of F, we get

$$B_n \supseteq \frac{1}{2^{n+1}}\left[\sum_{i=1}^{2^n}\left(1-\frac{i}{2^n}\right)F(a)+\frac{i}{2^n}F(b)+\left(1-\frac{i-1}{2^n}\right)F(a)+\frac{i-1}{2^n}F(b)\right]$$

$$= \frac{1}{2^{n+1}}\left[F(a)\sum_{i=1}^{2^n}\left(2-\frac{i}{2^{n-1}}+\frac{1}{2^n}\right)+F(b)\sum_{i=1}^{2^n}\left(\frac{i}{2^{n-1}}-\frac{1}{2^n}\right)\right]$$

$$= \frac{1}{2^{n+1}}\left[F(a)\left(2^{n+1}-\frac{1}{2^{n-1}}\cdot\frac{2^n\left(2^n+1\right)}{2}+\frac{2^n}{2^n}\right)\right.$$

$$\left.+F(b)\left(\frac{1}{2^{n-1}}\cdot\frac{2^n\left(2^n+1\right)}{2}-\frac{2^n}{2^n}\right)\right]$$

$$= \frac{1}{2^{n+1}}\left[F(a)\left(2^{n+1}-2^n\right)+F(b)\left(2^n\right)\right]=\frac{F(a)+F(b)}{2},$$

So

$$\frac{1}{(b-a)}\int_a^b F(x)dx \supseteq B_n \supseteq \frac{F(a)+F(b)}{2}.$$

On the other hand, by the left side of inclusion (1.2), we obtain

$$\frac{1}{(b-a)}\int_a^b F(x)dx = \frac{1}{b-a}\sum_{i=1}^{2^n}\int_{a+(i-1)\frac{b-a}{2^n}}^{a+i\frac{b-a}{2^n}}F(x)dx$$

$$\supseteq \frac{1}{b-a}\sum_{i=1}^{2^n}\frac{b-a}{2^n}F\left(\frac{a+i\frac{b-a}{2^n}+a+(i-1)\frac{b-a}{2^n}}{2}\right)$$

$$=\frac{1}{2^n}\sum_{i=1}^{2^n}F\left(a+i\frac{b-a}{2^n}-\frac{b-a}{2^{n+1}}\right)=A_n.$$

By utilizing the interval-valued convexity property of the function F and Jensen inclusion (2.3), we establish

$$A_n=\frac{1}{2^n}\sum_{i=1}^{2^n}F\left(a+i\frac{b-a}{2^n}-\frac{b-a}{2^{n+1}}\right)$$
$$\subseteq F\left[\frac{1}{2^n}\sum_{i=1}^{2^n}\left(a+i\frac{b-a}{2^n}-\frac{b-a}{2^{n+1}}\right)\right]$$
$$=F\left[\frac{1}{2^n}\left(2^n a+\frac{b-a}{2^n}\cdot\frac{2^n(2^n+1)}{2}-\frac{b-a}{2^{n+1}}2^n\right)\right]$$
$$=F\left(a+\frac{b-a}{2}\right)=F\left(\frac{a+b}{2}\right).$$

So, the proof is accomplished.

Remark 3.1. *Clearly, for $n=0$, Theorem 3.1 corresponds to (1.2).*
Corollary 3.2. *In Theorem 3.1, if we assign $n=1$, we obtain*

$$F\left(\frac{a+b}{2}\right)\supseteq\frac{1}{2}\left[F\left(\frac{3a+b}{4}\right)+F\left(\frac{a+3b}{4}\right)\right] \tag{3.1}$$

$$\supseteq\frac{1}{b-a}(IR)\int_a^b F(t)dt$$
$$\supseteq\frac{1}{4}\left[F(a)+2F\left(\frac{a+b}{2}\right)+F(b)\right]$$
$$\supseteq\frac{F(a)+F(b)}{2}.$$

Let's take the following example to demonstrate the correctness of the Corollary 3.2.

Example 3.3. *Define a function $F:[0,1]\to\mathbb{R}_I^+$ by $F(t)=[t^2,2-t^2]$, Then $F(t)$ is an interval-valued convex function such that $F(t)=[\underline{F}(t),\overline{F}(t)]$ for $t\in[0,1]$. By applying Corollary 3.2,* *the first expression of (3.1) becomes*

$$F\left(\frac{a+b}{2}\right)=F\left(\frac{1}{2}\right)=\left[\frac{1}{4},\frac{7}{4}\right], \tag{3.2}$$

The second expression of (3.1) becomes

$$\frac{1}{2}\left[F\left(\frac{3a+b}{4}\right)+F\left(\frac{a+3b}{4}\right)\right] \tag{3.3}$$

$$=\frac{1}{2}\left[\left[\frac{1}{16},\frac{31}{6}\right]+\left[\frac{9}{16},\frac{23}{16}\right]\right]=\left[\frac{5}{16},\frac{27}{16}\right].$$

By using definition of integral for interval valued function, the third expression of (3.1), we acquire

$$\frac{1}{b-a}(IR)\int_a^b F(t)dt=\left[\int_0^1 t^2dt,\int_0^1\left(2-t^2\right)dt\right]=\left[\frac{1}{3},\frac{5}{3}\right]. \tag{3.4}$$

And the fourth expression of (3.1), we obtain,

$$\frac{1}{4}\left[F(a)+2F\left(\frac{a+b}{2}\right)+F(b)\right] \tag{3.5}$$

$$=\frac{1}{4}\left[[0,2]+\left[\frac{1}{2},\frac{7}{2}\right]+[1,1]\right]=\left[\frac{3}{8},\frac{13}{8}\right].$$

And then fifth expression of (3.1), we get

$$\frac{F(a)+F(b)}{2}=\frac{[0,2]+[1,1]}{2}=\left[\frac{1}{2},\frac{3}{2}\right]. \tag{3.6}$$

Consequently,

$$\left[\frac{1}{4},\frac{7}{4}\right]\supseteq\left[\frac{5}{16},\frac{27}{16}\right]\supseteq\left[\frac{1}{3},\frac{5}{3}\right]\supseteq\left[\frac{3}{8},\frac{13}{8}\right]\supseteq\left[\frac{1}{2},\frac{3}{2}\right],$$

which demonstrates the result described in Corollary 3.2.

4. CONCLUSION

We have investigated a new extension of the Hermite–Hadamard inclusion in this paper. In this version we obtained, we showed that it turns into classic Hermite–Hadamard inclusion by special choices. In the other of these special choices, we got a new corollary. To demonstrate the correctness of the corollary, we have supplemented it with an example. In the future, the authors may examine fractional approaches to these inclusions. Also, interested researchers can obtain new inclusions with the help of different types of convexity. To the best of our knowledge, these results are new in the literature. We hope that the ideas and techniques of this paper will inspire interested readers working in this field. The ideas and techniques of this article may be the starting point for further research in this field.

REFERENCES

1. H. Budak, T. Tunç, M. Z. Sarikaya, Fractional Hermite-Hadamard-type inequalities for interval-valued functions, *Proceedings of the American Mathematical Society*, Vol 148, No 2, 705–718, (2020).
2. H. Budak, H. Kara, M. A. Ali, S. Khan and Y. Chu, Fractional Hermite–Hadamard type inequalities for interval-valued co-ordinated convex functions, *Open Mathematics*, Vol 19, No 1, 1081–1097, (2021).
3. H. Budak, H. Kara and S. Erden, On Fejér type Inequalities for products of Interval-valued convex functions, *Filomat*, 35(14), 4937–4955, (2021).
4. H. Budak and H. Kara, On the Weighted Generalization of Hermite–Hadamard Type Inclusions for Interval-Valued Convex Functions, *Filomat*, in press.
5. Y. Chalco-Cano, A. Flores-Franulic, H. Roman-Flores, Ostrowski type inequalities for interval-valued functions using generalized Hukuhara derivative, *Comput. Appl. Math.*, Vol 31, No 1, pp 457–472, (2012).
6. Y. Chalco-Cano,W. A. Lodwick,W. Condori-Equice, Ostrowski type inequalities and applications in numerical integration for interval-valued functions, *Soft Comput.*, Vol 19, No 1, pp 3293–3300, (2015).
7. T. M. Costa, Jensen's inequality type integral for fuzzy interval-valued functions, *Fuzzy Sets and Systems*, Vol 327, No 1, pp 31–47, (2017).
8. T. M. Costa, H. Roman-Flores, Some integral inequalities for fuzzy-interval-valued functions, *Inform. Sci.*, Vol 420, No 1, pp 110–125, (2017).
9. A. Dinghas, Zum Minkowskischen Integralbegriff abgeschlossener Mengen, *Math. Z.*, Vol 66, No 1, pp 173–188, (1956).
10. S. S. Dragomir and R.P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, *Appl. Math. lett.*, Vol 11, No 5, pp 91–95, (1998).
11. A. Flores-Franulic, Y. Chalco-Cano, H. Roman-Flores, An Ostrowski type inequality for interval-valued functions, *IFSA World Congress and NAFIPS Annual Meeting IEEE*, Vol 35, No 1, pp 1459–1462, (2013).
12. H. Roman-Flores, Y. Chalco-Cano, W. A. Lodwick, Some integral inequalities for interval-valued functions, *Comput. Appl. Math.*, Vol 37, No 1, pp 1306–1318, (2018).
13. H. Roman-Flores, Y. Chalco-Cano, G. N. Silva, A note on Gronwall type inequality for interval-valued functions, *IFSA World Congress and NAFIPS Annual Meeting IEEE*, Vol 35, No 1, pp 1455–1458, (2013).
14. R. Osuna-Gómez, M.D. Jiménez-Gamero, Y. Chalco-Cano, and M.A. Rojas-Medar, Hadamard and Jensen inequalities for s-convex fuzzy processes, *In: Soft Methodology and Random Information Systems, Springer, Berlin*, pp. 645–652, (2004).

15. H. Kara, M. A. Ali, and H. Budak, Hermite-Hadamard-type inequalities for interval-valued coordinated convex functions involving generalized fractional integrals, *Mathematical Methods in the Applied Sciences*, Vol 44, No 1, pp 104–123 (2021).
16. U.S. Kırmacı , Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, *Appl. Math. Comp.*, Vol 147, No 1, pp 137–146, (2004).
17. X. Liu, G. Ye, D. Zhao and W. Liu, Fractional Hermite–Hadamard type inequalities for interval-valued functions, *Journal of Inequalities and Applications*, Vol 1, No 1, pp 1–11, (2019).
18. F.C. Mitroi, N. Kazimierz, and W. Szymon, Hermite–Hadamard inequalities for convex set-valued functions, *Demonstratio Mathematica*, Vol XLVI, No 4, pp 655–662, (2013).
19. R. E. Moore, Interval analysis, *Prentice-Hall, Inc., Englewood Cliffs, N.J.*, (1966).
20. R. E. Moore, , R.B. Kearfott, M.J. Cloud, Introduction to interval analysis, *Society for Industrial and Applied Mathematics, Siam*, Vol 110, (2009).
21. K. Nikodem, On midpoint convex set-valued functions, *Aequationes Mathematicae*, Vol 33, No 1, pp 46–56, (1987).
22. K. Nikodem, J.L. Snchez, and L. Snchez, Jensen and Hermite-Hadamard inequalities for strongly convex set-valued maps, *Mathematica Aeterna*, Vol 4 No 8, pp 979–987, (2014).
23. B. Piatek, On the Riemann integral of set-valued functions, *Zeszyty Naukowe, Matematyka Stosowana/Politechnika Slaska*, (2012).
24. B. Piatek, On the Sincov functional equation, *Demonstratio Mathematica*, Vol 38, No 4, pp 875–882, (2005).
25. E. Sadowska, Hadamard inequality and a refinement of Jensen inequality for set-valued functions, *Results in Mathematics*, Vol 32, No. 1 pp 332–337, (1997).
26. G. Zabandan, *A new refinement of the Hermite-Hadamard inequality for convex functions*, J. Inequal. Pure Appl. Math, 10(2), 2009, 7.
27. D. Zhao, G. Ye, W. Liu, and D. F. Torres, Some inequalities for interval-valued functions on time scales, *Soft Computing*, Vol 23, No 15, pp 6005–6015, (2019).
28. D. Zhao, T. An, G. Ye, and W. Liu. Chebyshev type inequalities for interval-valued functions, *Fuzzy Sets and Systems*, Vol 396, No 1, pp 82–101, (2020).

Milne-Type Inequalities for Various Function Classes

Fatih HEZENCİ[1] and Hüseyin BUDAK[1]

[1] *Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, TÜRKİYE*

fatihezenci@gmail.com
hsyn.budak@gmail.com

ABSTRACT

The present abstract considers Milne-type inequalities for various function classes. Firstly, we give some Milne-type inequalities for bounded functions by fractional integrals. Moreover, we present several fractional Milne-type inequalities for the case of Lipschitzian functions.

1. INTRODUCTION

Fractional calculus is a branch of mathematics that investigates the properties of derivatives and integrals with fractional orders. It generalizes the classical calculus of integers, which is useful in in physics, engineering, and other fields. The commonly used definitions of fractional integrals include the Riemann-Liouville fractional, conformable fractional, and tempered fractional integrals, and more. The bounds of new formulas can be established by utilizing not only Hermite-Hadamard and Simpson-type inequalities but also Newton and Milne-type inequalities.
Let's introduce some initial concepts that will be utilized in the subsequent sections.

• Simpson's quadrature formula, commonly referred to as Simpson's $1/3$ rule, is formulated as follows:

$$\int_a^b f(x)dx \approx \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]. \tag{1}$$

• Simpson's second formula, also referred to as the Newton-Cotes quadrature formula or Simpson's $3/8$ rule (cf. [6]), is described as follows:

$$\int_a^b f(x)dx \approx \frac{b-a}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]. \tag{2}$$

Formulas (1) and (2) are suitable to any function f that includes a continuous fourth derivative on the closed interval $[a,b]$.
One of the most widely used Newton-Cotes quadrature methods including a three-point Simpson-type inequality is expressed as follows:

Theorem 1. *If $f:[a,b]\to\mathbb{R}$ is a four times continuously differentiable function on (a,b), and $\left\|f^{(4)}\right\|_\infty = \sup_{x\in(a,b)}\left|f^{(4)}(x)\right| < \infty$, then one has the following inequality*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\le\frac{1}{2880}\left\|f^{(4)}\right\|_\infty(b-a)^4.$$

Simpson-type inequalities and their application to quadrature inequalities in numerical analysis were presented in paper [1]. Moreover, in paper [2], some variants of Simpson-type inequalities are established for the case of differentiable convex functions by using generalized fractional integrals. Furthermore, in paper [3], fractional Simpson-type inequalities are investigated for the case of function whose second derivatives in absolute value are convex. For further information, please refer to Reference [4, 5] and the references cited within that source.

One of the classical closed-type quadrature rules is the Simpson $3/8$ rule, which is based on the Simpson $3/8$ inequality, formulated as follows:

Theorem 2 (See [6]). *Let us consider that* $f:[a,b]\to\mathbb{R}$ *is a four times continuously differentiable function on* (a,b), *and* $\left\|f^{(4)}\right\|_\infty=\sup_{x\in(a,b)}\left|f^{(4)}(x)\right|<\infty$. *Then, the following inequality holds:*

$$\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\le\frac{1}{6480}\left\|f^{(4)}\right\|_\infty(b-a)^4.$$

Simpson's second rule is a consequence of the three-point Newton-Cotes quadrature rule, leading to the frequent designation of evaluations including three-step quadratic kernels as Newton-type results. In the literature, these results are known as Newton-type inequalities. Considerable number of mathematicians have been investigated Newton-type inequalities. For example, some Newton-type inequalities were established for the case of functions whose first derivative in absolute value at certain power are arithmetically-harmonically convex in paper [7]. In addition, some Riemann-Liouville fractional Newton-type inequalities for functions of bounded variation were presented in paper [8]. Newton-type inequalities are further discussed in papers [9-11] and the referenced works within those papers provide additional information on this topic.
The open-type Milne formula, using Newton-Cotes formulas, is similar to the closed-type Simpson formula in that it is valid under the same conditions.

Theorem 3 (See [12]). *Let* $f:[a,b]\to\mathbb{R}$ *be a four times continuously differentiable mapping on* (a,b) *and* $\left\|f^{(4)}\right\|_\infty=\sup_{x\in(a,b)}\left|f^{(4)}(x)\right|<\infty$. *Then, one has the inequality*

$$\left|\frac{1}{3}\left[2f(a)-f\left(\frac{a+b}{2}\right)+2f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\le\frac{7(b-a)^4}{23040}\left\|f^{(4)}\right\|_\infty.$$

Ali et al. computed the error bounds by one of the open Newton–Cotes formulas, namely Milne's formula for differentiable convex functions in fractional and classical calculus in paper [13]. Ali et al. is proved integral equality to demonstrate the main findings as follows:

Lemma 1 (See [13]). *Let us consider that* $f:[a,b]\to\mathbb{R}$ *is an absolutely continuous function*

(a,b) *so that* $f' \in L_1[a,b]$. *Then, the following equality holds:*

$$\frac{1}{3}\left[2f\left(\frac{a+3b}{4}\right)-f\left(\frac{a+b}{2}\right)+2f\left(\frac{3a+b}{4}\right)\right]-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]$$
$$=\frac{b-a}{2}\sum_{i=1}^{4}I_i.$$

Here,

$$\begin{cases} I_1=\int_0^{\frac{1}{4}}t^{\alpha}\left[f'(tb+(1-t)a)-f'(ta+(1-t)b)\right]dt, \\ I_2=\int_{\frac{1}{4}}^{\frac{1}{2}}\left(t^{\alpha}-\frac{2}{3}\right)\left[f'(tb+(1-t)a)-f'(ta+(1-t)b)\right]dt, \\ I_3=\int_{\frac{1}{2}}^{\frac{3}{4}}\left(t^{\alpha}-\frac{1}{3}\right)\left[f'(tb+(1-t)a)-f'(ta+(1-t)b)\right]dt, \\ I_4=\int_{\frac{3}{4}}^{1}\left(t^{\alpha}-1\right)\left[f'(tb+(1-t)a)-f'(ta+(1-t)b)\right]dt. \end{cases}$$

The well-known Riemann-Liouville fractional integrals that are given as follows:

Definition 1 (See [14]). *The Riemann-Liouville integrals* $J_{a+}^{\alpha}f(x)$ *and* $J_{b-}^{\alpha}f(x)$ *of order* $\alpha>0$ *are presented by*

$$J_{a+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)dt,\ \ x>a$$

and

$$J_{b-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}f(t)dt,\ \ x<b,$$

respectively for $f\in L_1[a,b]$. The gamma function is defined $\Gamma(x):=\int_0^{\infty}t^{x-1}e^{-t}dt$ for $x\in\mathbb{R}^+$. The Riemann-Liouville integrals are equals to the classical integrals for the case of $\alpha=1$.
Djenaoui and Meftah [15] proved some estimates of Milne's quadrature rule for the case of functions whose first derivative is s-convex. In addition, Alomari and Liu [16] established error estimations for Milne's rule for functions of bounded variation and for absolutely continuous mappings. Moreover, Budak et al. [17] obtained fractional versions of Milne's formula by using the differentiable convex functions.

2. MILNE-TYPE INEQUALITIES FOR BOUNDED FUNCTIONS INVOLVING FRACTIONAL INTEGRALS

In this section, we present some Milne-type inequalities for bounded functions by using fractional integrals.

Theorem 4. *Suppose that the conditions of Lemma 1 hold. If there exist $m, M \in \mathbb{R}$ such that $m \leq f'(t) \leq M$ for $t \in [a,b]$. Then, we have*

$$\left|\frac{1}{3}\left[2f\left(\frac{a+3b}{4}\right)-f\left(\frac{a+b}{2}\right)+2f\left(\frac{3a+b}{4}\right)\right]-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\right| \tag{3}$$
$$\leq \frac{b-a}{2}(M-m)\{\Omega_1(\alpha)+\Omega_2(\alpha)+\Omega_3(\alpha)+\Omega_4(\alpha)\},$$

where

$$\Omega_1(\alpha)=\int_0^{\frac{1}{4}} t^{\alpha}dt=\frac{1}{4^{\alpha+1}(\alpha+1)},$$

$$\Omega_2(\alpha)=\int_{\frac{1}{4}}^{\frac{1}{2}}\left|t^{\alpha}-\frac{2}{3}\right|dt=\begin{cases} -\dfrac{\left((\alpha+1)2^{\alpha+1}-6\right)4^{\alpha}+2^{\alpha}3}{12(\alpha+1)2^{\alpha}4^{\alpha}}, & 0<\alpha\leq\dfrac{\ln\left(\frac{2}{3}\right)}{\ln\left(\frac{1}{4}\right)}, \\[2ex] -\dfrac{\left(2(\alpha+1)3^{\frac{1}{\alpha}}-\alpha 2^{\frac{1}{\alpha}+3}\right)4^{\alpha}-3^{\frac{1}{\alpha}+1}}{12(\alpha+1)3^{\frac{1}{\alpha}}4^{\alpha}} -\dfrac{\left(2(\alpha+1)3^{\frac{1}{\alpha}}-\alpha 2^{\frac{1}{\alpha}+2}\right)2^{\alpha}-3^{\frac{1}{\alpha}+1}}{6(\alpha+1)3^{\frac{1}{\alpha}}2^{\alpha}}, & \dfrac{\ln\left(\frac{2}{3}\right)}{\ln\left(\frac{1}{4}\right)}<\alpha\leq\dfrac{\ln\left(\frac{2}{3}\right)}{\ln\left(\frac{1}{2}\right)}, \\[2ex] \dfrac{\left((\alpha+1)2^{\alpha+1}-6\right)4^{\alpha}+2^{\alpha}3}{12(\alpha+1)2^{\alpha}4^{\alpha}}, & \alpha>\dfrac{\ln\left(\frac{2}{3}\right)}{\ln\left(\frac{1}{2}\right)}, \end{cases}$$

$$\Omega_3(\alpha)=\int_{\frac{1}{2}}^{\frac{3}{4}}\left|t^{\alpha}-\frac{1}{3}\right|dt=\begin{cases}\dfrac{3^{\alpha+1}}{4^{\alpha+1}(\alpha+1)}-\dfrac{1}{2^{\alpha+1}(\alpha+1)}-\dfrac{1}{12}, & 0<\alpha\le\dfrac{\ln\left(\frac{1}{3}\right)}{\ln\left(\frac{1}{2}\right)},\\[2ex] \dfrac{3^{\frac{1}{\alpha}+1}-\left((\alpha+1)3^{\frac{1}{\alpha}}-2\alpha\right)2^{\alpha}}{6(\alpha+1)3^{\frac{1}{\alpha}}2^{\alpha}} \\ -\dfrac{\left(3(\alpha+1)3^{\frac{1}{\alpha}}-4\alpha\right)4^{\alpha}-3^{\alpha+\frac{1}{\alpha}+2}}{12(\alpha+1)3^{\frac{1}{\alpha}}4^{\alpha}}, & \dfrac{\ln\left(\frac{1}{3}\right)}{\ln\left(\frac{1}{2}\right)}<\alpha\le\dfrac{\ln\left(\frac{1}{3}\right)}{\ln\left(\frac{3}{4}\right)},\\[2ex] \dfrac{1}{2^{\alpha+1}(\alpha+1)}-\dfrac{3^{\alpha+1}}{4^{\alpha+1}(\alpha+1)}+\dfrac{1}{12}, & \alpha>\dfrac{\ln\left(\frac{1}{3}\right)}{\ln\left(\frac{3}{4}\right)},\end{cases}$$

and

$$\Omega_4(\alpha)=\int_{\frac{3}{4}}^{1}\left|t^{\alpha}-1\right|dt=\frac{\alpha-3}{4(\alpha+1)}+\frac{1}{\alpha+1}\left(\frac{3}{4}\right)^{\alpha+1}.$$

Proof. Let us first consider the Lemma 1. Then, we easily get

$$\frac{1}{3}\left[2f\left(\frac{a+3b}{4}\right)-f\left(\frac{a+b}{2}\right)+2f\left(\frac{3a+b}{4}\right)\right]-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right] \tag{4}$$

$$=\frac{b-a}{2}\left\{\int_0^{\frac{1}{4}}t^{\alpha}\left[f'(tb+(1-t)a)-\frac{m+M}{2}\right]dt\right.$$

$$+\int_0^{\frac{1}{4}}t^{\alpha}\left[\frac{m+M}{2}-f'(ta+(1-t)b)\right]dt$$

$$+\int_{\frac{1}{4}}^{\frac{1}{2}}\left(t^{\alpha}-\frac{2}{3}\right)\left[f'(tb+(1-t)a)-\frac{m+M}{2}\right]dt$$

$$+\int_{\frac{1}{4}}^{\frac{1}{2}}\left(t^{\alpha}-\frac{2}{3}\right)\left[\frac{m+M}{2}-f'(ta+(1-t)b)\right]dt$$

$$+\int_{\frac{1}{2}}^{\frac{3}{4}}\left(t^{\alpha}-\frac{1}{3}\right)\left[f'\left(tb+(1-t)a\right)-\frac{m+M}{2}\right]dt$$

$$+\int_{\frac{1}{2}}^{\frac{3}{4}}\left(t^{\alpha}-\frac{1}{3}\right)\left[\frac{m+M}{2}-f'\left(ta+(1-t)b\right)\right]dt$$

$$+\int_{\frac{3}{4}}^{1}\left(t^{\alpha}-1\right)\left[f'\left(tb+(1-t)a\right)-\frac{m+M}{2}\right]dt$$

$$\left.+\int_{\frac{3}{4}}^{1}\left(t^{\alpha}-1\right)\left[\frac{m+M}{2}-f'\left(ta+(1-t)b\right)\right]dt\right\}.$$

By using the absolute value of (4), we obtain

$$\left|\frac{1}{3}\left[2f\left(\frac{a+3b}{4}\right)-f\left(\frac{a+b}{2}\right)+2f\left(\frac{3a+b}{4}\right)\right]-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\right|$$

$$\leq\frac{b-a}{2}\left\{\int_{0}^{\frac{1}{4}}t^{\alpha}\left|f'\left(tb+(1-t)a\right)-\frac{m+M}{2}\right|dt\right.$$

$$+\int_{0}^{\frac{1}{4}}t^{\alpha}\left|\frac{m+M}{2}-f'\left(ta+(1-t)b\right)\right|dt$$

$$+\int_{\frac{1}{4}}^{\frac{1}{2}}\left|t^{\alpha}-\frac{2}{3}\right|\left|f'\left(tb+(1-t)a\right)-\frac{m+M}{2}\right|dt$$

$$+\int_{\frac{1}{4}}^{\frac{1}{2}}\left|t^{\alpha}-\frac{2}{3}\right|\left|\frac{m+M}{2}-f'\left(ta+(1-t)b\right)\right|dt$$

$$+\int_{\frac{1}{2}}^{\frac{3}{4}}\left|t^{\alpha}-\frac{1}{3}\right|\left|f'\left(tb+(1-t)a\right)-\frac{m+M}{2}\right|dt$$

$$+\int_{\frac{1}{2}}^{\frac{3}{4}}\left|t^{\alpha}-\frac{1}{3}\right|\left|\frac{m+M}{2}-f'\left(ta+(1-t)b\right)\right|dt$$

$$+\int_{\frac{3}{4}}^{1}\left|t^{\alpha}-1\right|\left|f'(tb+(1-t)a)-\frac{m+M}{2}\right|dt$$
$$\left.+\int_{\frac{3}{4}}^{1}\left|t^{\alpha}-1\right|\left|\frac{m+M}{2}-f'(ta+(1-t)b)\right|dt\right\}.$$

It is known that $m\leq f'(t)\leq M$ for $t\in[a,b]$. Then, we have

$$\left|f'(tb+(1-t)a)-\frac{m+M}{2}\right|\leq\frac{M-m}{2} \tag{5}$$

and

$$\left|\frac{m+M}{2}-f'(ta+(1-t)b)\right|\leq\frac{M-m}{2}. \tag{6}$$

By using (5) and (6), we get

$$\left|\frac{1}{3}\left[2f\left(\frac{a+3b}{4}\right)-f\left(\frac{a+b}{2}\right)+2f\left(\frac{3a+b}{4}\right)\right]-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\right|$$
$$\leq\frac{b-a}{2}(M-m)\left\{\int_{0}^{\frac{1}{4}}t^{\alpha}dt+\int_{\frac{1}{4}}^{\frac{1}{2}}\left|t^{\alpha}-\frac{2}{3}\right|dt+\int_{\frac{1}{2}}^{\frac{3}{4}}\left|t^{\alpha}-\frac{1}{3}\right|dt+\int_{\frac{3}{4}}^{1}\left|t^{\alpha}-1\right|dt\right\}$$
$$=\frac{b-a}{2}(M-m)\{\Omega_1(\alpha)+\Omega_2(\alpha)+\Omega_3(\alpha)+\Omega_4(\alpha)\}.$$

Corollary 1. *If we choose* $\alpha=1$ *in Theorem 4, then we obtain*

$$\left|\frac{1}{3}\left[2f\left(\frac{a+3b}{4}\right)-f\left(\frac{a+b}{2}\right)+2f\left(\frac{3a+b}{4}\right)\right]-\frac{1}{b-a}\int_{a}^{b}f(t)dt\right|$$
$$\leq\frac{5(b-a)}{48}(M-m).$$

Corollary 2. *Under assumption of Theorem 4, if there exist* $M\in\mathbb{R}^{+}$ *such that* $|f'(t)|\leq M$ *for all* $t\in[a,b]$, *then we have*

$$\left|\frac{1}{3}\left[2f\left(\frac{a+3b}{4}\right)-f\left(\frac{a+b}{2}\right)+2f\left(\frac{3a+b}{4}\right)\right]-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\right|$$
$$\leq(b-a)M\{\Omega_1(\alpha)+\Omega_2(\alpha)+\Omega_3(\alpha)+\Omega_4(\alpha)\}.$$

Corollary 3. *If we assign* $\alpha=1$ *in Corollary 2, then the following inequality holds:*

$$\left|\frac{1}{3}\left[2f\left(\frac{a+3b}{4}\right)-f\left(\frac{a+b}{2}\right)+2f\left(\frac{3a+b}{4}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{5(b-a)}{24}M.$$

3. FRACTIONAL MILNE-TYPE INEQUALITIES BY USING LIPSCHITZIAN FUNCTIONS

In this section, we present some fractional Milne-type inequalities for Lipschitzian functions.

Theorem 5. *Assume that the assumptions of Lemma 1 are valid. If f' is a L-Lipschitzian function on $[a,b]$, then the following inequality*

$$\left|\frac{1}{3}\left[2f\left(\frac{a+3b}{4}\right)-f\left(\frac{a+b}{2}\right)+2f\left(\frac{3a+b}{4}\right)\right]-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\right|$$

$$\leq\frac{(b-a)^2}{2}L\{\Omega_5(\alpha)+\Omega_2(\alpha)-2\Omega_6(\alpha)+2\Omega_3(\alpha)-\Omega_7(\alpha)+2\Omega_4(\alpha)-\Omega_8(\alpha)\}.$$

is valid. Here,

$$\Omega_5(\alpha)=\int_0^{\frac{1}{4}}t^{\alpha}(1-2t)dt=\frac{1}{4^{\alpha+1}(\alpha+1)}-\frac{2}{4^{\alpha+2}(\alpha+2)},$$

$$\Omega_6(\alpha)=\int_{\frac{1}{4}}^{\frac{1}{2}}t\left|t^{\alpha}-\frac{2}{3}\right|dt=\begin{cases}\frac{1}{\alpha+2}\left(\frac{1}{2}\right)^{\alpha+2}-\frac{1}{\alpha+2}\left(\frac{1}{4}\right)^{\alpha+2}-\frac{1}{16}, & 0<\alpha\leq\frac{\ln\left(\frac{2}{3}\right)}{\ln\left(\frac{1}{4}\right)},\\[2ex] \frac{\alpha}{\alpha+2}\left(\frac{2}{3}\right)^{1+\frac{2}{\alpha}}+\frac{1}{\alpha+2}\left(\frac{1}{4}\right)^{\alpha+2}+\frac{1}{\alpha+2}\left(\frac{1}{2}\right)^{\alpha+2}-\frac{5}{48}, & \frac{\ln\left(\frac{2}{3}\right)}{\ln\left(\frac{1}{4}\right)}<\alpha\leq\frac{\ln\left(\frac{2}{3}\right)}{\ln\left(\frac{1}{2}\right)},\\[2ex] -\frac{1}{\alpha+2}\left(\frac{1}{2}\right)^{\alpha+2}+\frac{1}{\alpha+2}\left(\frac{1}{4}\right)^{\alpha+2}+\frac{1}{16}, & \alpha>\frac{\ln\left(\frac{2}{3}\right)}{\ln\left(\frac{1}{2}\right)},\end{cases}$$

$$\Omega_7(\alpha)=\int_{\frac{1}{2}}^{\frac{3}{4}} t\left|t^{\alpha}-\frac{1}{3}\right|dt=\begin{cases}\frac{1}{\alpha+2}\left(\frac{3}{4}\right)^{\alpha+2}-\frac{1}{\alpha+2}\left(\frac{1}{2}\right)^{\alpha+2}-\frac{5}{96}, & 0<\alpha\leq\frac{\ln\left(\frac{1}{3}\right)}{\ln\left(\frac{1}{2}\right)},\\[2ex] \frac{\alpha}{\alpha+2}\left(\frac{1}{3}\right)^{1+\frac{2}{\alpha}}+\frac{1}{\alpha+2}\left(\frac{1}{2}\right)^{\alpha+2} & \frac{\ln\left(\frac{1}{3}\right)}{\ln\left(\frac{1}{2}\right)}<\alpha\leq\frac{\ln\left(\frac{1}{3}\right)}{\ln\left(\frac{3}{4}\right)},\\ +\frac{1}{\alpha+2}\left(\frac{3}{4}\right)^{\alpha+2}-\frac{13}{96}, & \\[2ex] -\frac{1}{\alpha+2}\left(\frac{3}{4}\right)^{\alpha+2}+\frac{1}{\alpha+2}\left(\frac{1}{2}\right)^{\alpha+2}+\frac{5}{96}, & \alpha>\frac{\ln\left(\frac{1}{3}\right)}{\ln\left(\frac{3}{4}\right)},\end{cases}$$

and

$$\Omega_8(\alpha)=\int_{\frac{3}{4}}^{1} t\left|t^{\alpha}-1\right|dt=\frac{7}{32}-\frac{1}{\alpha+2}+\frac{1}{\alpha+2}\left(\frac{3}{4}\right)^{\alpha+2}.$$

Proof. Let us consider Lemma 1 and f' is L-Lipschitzian function. Then, we obtain

$$\begin{aligned}&\left|\frac{1}{3}\left[2f\left(\frac{a+3b}{4}\right)-f\left(\frac{a+b}{2}\right)+2f\left(\frac{3a+b}{4}\right)\right]-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\right|\\ &=\left|\frac{b-a}{2}\left\{\int_0^{\frac{1}{4}} t^{\alpha}\left[f'(tb+(1-t)a)-f'(ta+(1-t)b)\right]dt\right.\right.\\ &\quad+\int_{\frac{1}{4}}^{\frac{1}{2}}\left(t^{\alpha}-\frac{2}{3}\right)\left[f'(tb+(1-t)a)-f'(ta+(1-t)b)\right]dt\\ &\quad+\int_{\frac{1}{2}}^{\frac{3}{4}}\left(t^{\alpha}-\frac{1}{3}\right)\left[f'(tb+(1-t)a)-f'(ta+(1-t)b)\right]dt\\ &\quad\left.\left.+\int_{\frac{3}{4}}^{1}\left(t^{\alpha}-1\right)\left[f'(tb+(1-t)a)-f'(ta+(1-t)b)\right]dt\right\}\right|\\ &\leq\frac{b-a}{2}\left\{\int_0^{\frac{1}{4}} t^{\alpha}\left|f'(tb+(1-t)a)-f'(ta+(1-t)b)\right|dt\right.\end{aligned}$$

$$+\int_{\frac{1}{4}}^{\frac{1}{2}}\left|t^{\alpha}-\frac{2}{3}\right|\left|f'(tb+(1-t)a)-f'(ta+(1-t)b)\right|dt$$

$$+\int_{\frac{1}{2}}^{\frac{3}{4}}\left|t^{\alpha}-\frac{1}{3}\right|\left|f'(tb+(1-t)a)-f'(ta+(1-t)b)\right|dt$$

$$\left.+\int_{\frac{3}{4}}^{1}(1-t^{\alpha})\left|f'(tb+(1-t)a)-f'(ta+(1-t)b)\right|dt\right\}$$

$$\leq \frac{b-a}{2}\left\{\int_0^{\frac{1}{4}} t^{\alpha}L(1-2t)(b-a)dt+\int_{\frac{1}{4}}^{\frac{1}{2}}\left|t^{\alpha}-\frac{2}{3}\right|L(1-2t)(b-a)dt\right.$$

$$\left.+\int_{\frac{1}{2}}^{\frac{3}{4}}\left|t^{\alpha}-\frac{1}{3}\right|L(2t-1)(b-a)dt+\int_{\frac{3}{4}}^{1}(1-t^{\alpha})L(2t-1)(b-a)dt\right\}$$

$$=\frac{(b-a)^2}{2}L\{\Omega_5(\alpha)+\Omega_2(\alpha)-2\Omega_6(\alpha)+2\Omega_3(\alpha)-\Omega_7(\alpha)+2\Omega_4(\alpha)-\Omega_8(\alpha)\}.$$

Corollary 4. *If we choose* $\alpha=1$ *in Theorem 5, then the following Milne-type inequality holds:*

$$\left|\frac{1}{3}\left[2f\left(\frac{a+3b}{4}\right)-f\left(\frac{a+b}{2}\right)+2f\left(\frac{3a+b}{4}\right)\right]-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{(b-a)^2}{24}L.$$

5. CONCLUSION

The present extended abstract considers Milne-type inequalities for various function classes. Firstly, we present several Milne-type inequalities for bounded functions by fractional integrals. Moreover, we present several fractional Milne-type inequalities for the case of Lipschitzian functions.
In future investigations, the exploration of concepts and strategies connected to our findings regarding Milne-type inequalities through Riemann-Liouville fractional integrals has the potential to pave the way for innovative pathways in the field of mathematics. Moreover, one may consider generalizing our findings by exploring alternative classes of convex functions or different types of fractional integral operators. Moreover, one can obtain Milne-type inequalities by Riemann-Liouville fractional integrals for convex functions by using quantum calculus.

REFERENCES

1. Dragomir S.S.; Agarwal R.P.; Cerone P. On Simpson's inequality and applications, *J. Inequal. Appl.*, 5, 533–579, 2000.
2. Budak H.; Hezenci F.; Kara H. On parametrized inequalities of Ostrowski and Simpson type for convex functions via generalized fractional integral, *Math. Methods Appl. Sci.*, 44(30), 12522–12536, 2021.

3. Hezenci, F. A note on Fractional Simpson type inequalities for twice differentiable functions, *Mathematica Slovaca*,73 (3), 675–686, 2021.
4. M. R. Delavar and M. De La Sen, On generalization of Fejér type inequalities, Communications in Mathematics and Applications, vol. 8, no. 1, pp. 31--43, 2017.
5. Hezenci, F.; Budak, H.; and Kara, H. New version of Fractional Simpson type inequalities for twice differentiable functions, *Advances in Difference Equations*, 2021(460), 2021.
6. Davis P.J.; Rabinowitz P.; *Methods of numerical integration*, Academic Press, New York-San Francisco-London, 1975.
7. Erden, S.; Iftikhar, S.; Kumam, P.; and Awan, M. U. Some Newton's like inequalities with applications, *Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas*, 114(4), 1–13, 2020.
8. Hezenci, F.; Budak, H.; Kosem, P. A new version of Newton's inequalities for Riemann-Liouville fractional integrals, *Rocky Mountain Journal of Mathematics*, 53(1) (2023) 49–64.
9. Hezenci, F.; Budak, H. Some Perturbed Newton type inequalities for Riemann-Liouville fractional integrals, *Rocky Mountain Journal of Mathematics*, 53(4) (2023) 1117–1127.
10. Iftikhar, S.; Erden, S.; Kumam, P.; and Awan, M.U. Local fractional Newton's inequalities involving generalized harmonic convex functions, *Advances in Difference Equations*, 2020(1), 1–14, 2020.
11. Noor, M.A.; Noor, K.I.; Iftikhar, S. Some Newton's type inequalities for harmonic convex functions, *J. Adv. Math. Stud.*, 9(1), 07–16, 2016.
12. Booth, A. D., (1966). Numerical methods, 3rd Ed., Butterworths, California.
13. Ali M.A., Zhang Z., Fečkan M. On Some Error Bounds for Milne's Formula in Fractional Calculus. Mathematics. 2023; 11(1):146. https://doi.org/10.3390/math11010146.
14. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204, Elsevier Sci. B.V., Amsterdam, 2006.
15. Djenaoui, M. and Meftah, B. (2022). Milne type inequalities for differentiable s-convex functions, Honam Mathematical J. 44, No. 3, pp. 325-338.
16. Alomari, M. (2013). New error estimations for the Milne's quadrature formula in terms of at most first derivatives, Konuralp J. Math., 1(1), 2013, pp. 17-23.
17. Budak, H., Kösem P., and Kara, H., On new Milne-type inequalities for fractional integrals, Journal of Inequalities and Applications, 2023(10), 2023.

Generalizations of Different Type Inequalities for η-Convex Function

Barış ÇELİK[1], Erhan SET[1] and Alper EKİNCİ[2]

[1]*Department of Mathematics, Faculty of Science and Arts,*
Ordu University, Ordu-TURKEY
[2]*Bandırma Vocational School,*
Bandırma Onyedi Eylül University, Balikesir-TURKEY

bariscelik15@hotmail.com
erhanset@yahoo.com
alperekinci@hotmail.com

ABSTRACT

Several extensions, generalizations and new variant of different types of inequalities for different kinds of convex functions obtained by researchers. In this paper, we establish the Bullen, Midpoint, Trapezoid and Simpson type inequalities, respectively, for η-convex function, with the help of identities existing in the literature.

INTRODUCTION

Convexity theory is an astonishing and compelling methodology for contemplating the enormous and beautiful issues that arise in many different fields of the pure and applied sciences. Numerous new structures have been presented and explored, including convex sets and related functions. This theory has a rich history and has been the focus and motivation of outstanding mathematical research for more than a century. Also, convexity theory has a critical place in the advancement of the idea of inequality. Inequalities have an interesting mathematical model due to their important applications in traditional calculus, fractional calculus, quantum calculus, interval-valued, stochastic, time-scale calculus, fractal sets, etc.

One of the functions defined in the class of convex functions is the η-convex function. In [9], Gordji et al. introduced the idea of η-convex functions as generalization of ordinary convex functions and gave the following definition for η-convexity of functions.

Definition 1 *A function $f:[a,b]\to\mathbb{R}$ is said to be $\eta-$convex (or convex with respect to η) if the inequality*

$$f(tx+(1-t)y)\le f(y)+t\eta(f(x),f(y))$$

holds for all $x,y\in[a,b]$, $t\in[0,1]$ and η is defined by $\eta: f([a,b])\times f([a,b])\to\mathbb{R}$.

In the above definition if we set $\eta(x,y)=x-y$, then we can directly obtain the classical definition of a convex function. Too see more results and details on η-convex functions see [3, 10, 11].

Recently, a large number of researchers, including mathematicians, engineers and scientists, have devoted themselves to studying the inequalities and properties associated with convexity in certain different directions. Many integral inequalities have been developed so far by different researchers in the due course of time. In the literature, we have many types of inequalities that involve convex functions, such as Bullen inequality [2], Hermite-Hadamard-Fejer inequality [8], Simpson type inequality [18], and Ostrowski type inequalities [17]. Likewise, there are a lot of well-known integral inequalities but the most notable one is the Hermite-Hadamard integral inequality.

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an integrable convex function with $a < b$. Then, the Hermite-Hadamard inequality is expressed as follows: (see [12]):

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

In [9], Gordji et al. proved some important results but here we give only one of them in the following theorem based on the above definition, which is also known as η-convex version of Hermite-Hadamard inequality.

Theorem 1 *[9] Suppose that* $f : [a,b] \to \mathbb{R}$ *is a* $\eta-$*convex function such that* η *is bounded above on* $f([a,b]) \times f([a,b])$. *Then the following inequalities hold.*

$$f\left(\frac{a+b}{2}\right) - \frac{M_\eta}{2} \leq \frac{1}{b-a}\int_a^b f(x)dx$$

$$\leq \frac{1}{2}[f(a)+f(b)] + \frac{1}{4}[\eta(f(a), f(b)) + \eta(f(b), f(a))] \tag{1.1}$$

$$\leq \frac{f(a)+f(b)}{2} + \frac{M_\eta}{2}, \tag{1.2}$$

where M_η is the upper bound of η.

There has also been research focusing on the Simpson-type inequality. Many researchers have studied Simpson-type inequalities in the literature (see, [1, 4, 7, 13, 14]).

Bullen [2] obtained the well-known Bullen-type inequalities. Bullen-type inequalities for generalized convex functions were obtained by SarÄ±kaya and Budak [16]. The local fractional version of Bullen-type inequality were presented in [6]. Du *et al.* [5] obtained Bullen-type inequalities using fractional integrals.

In the last few decades, many mathematicians and research scholars have focused their great contributions and attention to the study of this inequality. The aim of this paper, is to establish some new Hermite-Hadamard type inequalities and Simpson-type inequalities for η-convex function.

GENERALIZED BULLEN TYPE INEQUALITIES

Theorem 2 *Suppose that* $f:[0,\infty)\to[0,\infty)$ *is an* η*-convex function, where let* $a,b\in[0,\infty)$, $a<b$. *If* $f\in L[a,b]$, *then the following inequalities hold*

$$f\left(\frac{a+x}{2}\right)+f\left(\frac{b+x}{2}\right)-M_\eta\le\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt$$

$$\le\frac{f(a)+f(b)+2f(x)}{2}+M_\eta \tag{2.1}$$

for $x\in(a,b)$.

Proof. Since f is a η-convex function on $[a,x]\subset[a,b]$, by using the inequalities (1.1) we get

$$f\left(\frac{a+x}{2}\right)-\frac{M_\eta}{2}\le\frac{1}{x-a}\int_a^x f(t)dt\le\frac{f(a)+f(x)}{2}+\frac{M_\eta}{2}. \tag{2.2}$$

By similar way for $[x,b]\subset[a,b]$, it follows that

$$f\left(\frac{b+x}{2}\right)-\frac{M_\eta}{2}\le\frac{1}{b-a}\int_x^b f(t)dt\le\frac{f(b)+f(x)}{2}+\frac{M_\eta}{2}. \tag{2.3}$$

Consequently, by adding (2.2) and (2.3), we have

$$f\left(\frac{a+x}{2}\right)+f\left(\frac{b+x}{2}\right)-M_\eta\le\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt$$

$$\le\frac{f(a)+f(b)+2f(x)}{2}+M_\eta$$

which completes the proof of (2.1).

TRAPEZOID TYPE INEQUALITIES

Lemma 1 *[15] Let* $f:[a,b]\to\mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $a<b$. *If* $f'\in L[a,b]$, *then the following equality holds:*

$$f(x)+\frac{f(a)+f(b)}{2}-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]$$

$$=\frac{x-a}{2}\int_0^1(1-2\lambda)f'(\lambda a+(1-\lambda)x)d\lambda$$

$$+\frac{b-x}{2}\int_0^1(1-2\lambda)f'(\lambda x+(1-\lambda)b)d\lambda$$

for $x\in[a,b]$.

Theorem 3 *Let* $f:[a,b]\subset[0,\infty)\to\mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $a<b$. *If* $|f'|$ *is* η*-convex, then*

$$\left|f(x)+\frac{f(a)+f(b)}{2}-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right| \tag{3.1}$$

$$\le\frac{(x-a)|f'(x)|+(b-x)|f'(b)|}{4}$$

$$+\frac{(x-a)\eta(|f'(a)|,|f'(x)|)+(b-x)\eta(|f'(x)|,|f'(b)|)}{8}$$

for $x\in[a,b]$.

Proof. From Lemma 1, by using the properties of modulus and $|f'|$ is η-convex, we have

$$\left|f(x)+\frac{f(a)+f(b)}{2}-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right|$$

$$\leq\frac{x-a}{2}\int_0^1|1-2\lambda||f'(\lambda a+(1-\lambda)x)d\lambda|$$

$$+\frac{b-x}{2}\int_0^1|1-2\lambda||f'(\lambda x+(1-\lambda)b)d\lambda|$$

$$\leq\frac{x-a}{2}\int_0^1|1-2\lambda|\left[|f'(x)|+\lambda\eta(|f'(a)|,|f'(x)|)\right]d\lambda$$

$$+\frac{b-x}{2}\int_0^1|1-2\lambda|\left[|f'(b)|+\lambda\eta(|f'(x)|,|f'(b)|)\right]d\lambda$$

$$=\frac{x-a}{2}\int_0^{\frac{1}{2}}(1-2\lambda)\left[|f'(x)|+\lambda\eta(|f'(a)|,|f'(x)|)\right]d\lambda$$

$$+\frac{x-a}{2}\int_{\frac{1}{2}}^1(2\lambda-1)\left[|f'(x)|+\lambda\eta(|f'(a)|,|f'(x)|\right]d\lambda$$

$$+\frac{b-x}{2}\int_0^{\frac{1}{2}}(1-2\lambda)\left[|f'(b)|+\lambda\eta(|f'(x)|,|f'(b)|)\right]d\lambda$$

$$+\frac{b-x}{2}\int_{\frac{1}{2}}^1(2\lambda-1)\left[|f'(b)|+\lambda\eta(|f'(x)|,|f'(b)|)\right]d\lambda$$

$$=\frac{(x-a)|f'(x)|+(b-x)|f'(b)|}{4}$$

$$+\frac{(x-a)\eta(|f'(a)|,|f'(x)|)+(b-x)\eta(|f'(x)|,|f'(b)|)}{8}$$

which completes the proof of the inequality (3.1).

Theorem 4 *Let* $f:[a,b]\subset[0,\infty)\to\mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $a<b$. *If* $|f'|^q$ *is* η*-convex for some* $q>1$, *then*

$$\left|f(x)+\frac{f(a)+f(b)}{2}-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right| \tag{3.2}$$

$$\leq\frac{x-a}{2(p+1)^{\frac{1}{p}}}\left(|f'(x)|^q+\frac{\eta(|f'(a)|^q+|f'(x)|^q)}{2}\right)^{\frac{1}{q}}$$

$$+\frac{b-x}{2(p+1)^{\frac{1}{p}}}\left(|f'(b)|^q+\frac{\eta(|f'(x)|^q+|f'(b)|^q)}{2}\right)^{\frac{1}{q}}$$

where $x\in[a,b]$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 1, by using Hölder inequality and $|f'|^q$ is η-convex, we have

$$\left|f(x)+\frac{f(a)+f(b)}{2}-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right|$$
$$\leq\frac{x-a}{2}\int_0^1|1-2\lambda||f'(\lambda a+(1-\lambda)x)d\lambda|$$
$$+\frac{b-x}{2}\int_0^1|1-2\lambda||f'(\lambda x+(1-\lambda)b)d\lambda|$$
$$\leq\frac{x-a}{2}\left(\int_0^1|1-2\lambda|^p\,d\lambda\right)^{\frac{1}{p}}\left(\int_0^1|f'(\lambda a+(1-\lambda)x)d\lambda|\right)^{\frac{1}{q}}$$
$$+\frac{b-x}{2}\left(\int_0^1|1-2\lambda|^p\,d\lambda\right)^{\frac{1}{p}}\left(\int_0^1|f'(\lambda x+(1-\lambda)b)d\lambda|\right)^{\frac{1}{q}}$$
$$\leq\frac{x-a}{2(p+1)^{\frac{1}{p}}}\left(\int_0^1(|f'(x)|^q+\lambda\eta(|f'(a)|^q,|f'(x)|^q)d\lambda\right)^{\frac{1}{q}}$$
$$+\frac{b-x}{2(p+1)^{\frac{1}{p}}}\left(\int_0^1(|f'(b)|^q+\lambda\eta(|f'(x)|^q,|f'(b)|^q)d\lambda\right)^{\frac{1}{q}}$$
$$=\frac{x-a}{2(p+1)^{\frac{1}{p}}}\left(|f'(x)|^q+\frac{\eta(|f'(a)|^q+|f'(x)|^q)}{2}\right)^{\frac{1}{q}}$$
$$+\frac{b-x}{2(p+1)^{\frac{1}{p}}}\left(|f'(b)|^q+\frac{\eta(|f'(x)|^q+|f'(b)|^q)}{2}\right)^{\frac{1}{q}}$$

which completes the proof of the inequality (3.2).

MIDPOINT TYPE INEQUALITIES

Lemma 2 *[15] Let $f:[a,b]\to\mathrm{R}$ be a differentiable mapping on (a,b) with $a<b$. $f'\in L[a,b]$, then the following equality holds:*

$$f\left(\frac{a+x}{2}\right)+f\left(\frac{b+x}{2}\right)-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]$$
$$=(x-a)\int_0^{\frac{1}{2}}\lambda[f'(\lambda x+(1-\lambda)a)-f'(\lambda a+(1-\lambda)x)]d\lambda$$
$$+(b-x)\int_{\frac{1}{2}}^1(1-\lambda)[f'(\lambda x+(1-\lambda)b)-f'(\lambda b+(1-\lambda)x)]d\lambda$$

for $x\in[a,b]$.

Theorem 5 *Let* $f:[a,b]\subset[0,\infty)\to\mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $a<b$. *If* $|f'|$ *is* η*-convex, then*

$$\left|f\left(\frac{a+x}{2}\right)+f\left(\frac{b+x}{2}\right)-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right| \tag{4.1}$$
$$\leq\frac{(x-a)(|f'(a)|+|f'(x)|)+(b-x)(|f'(b)|+|f'(x)|)}{8}$$
$$+\frac{(x-a)(\eta(|f'(x)|,|f'(a)|))+(b-x)(\eta(|f'(b)|,|f'(x)|))}{12}$$

for $x\in[a,b]$.

Proof. From Lemma 2, by using the properties of modulus and $|f'|$ is η-convex, we have

$$\left|f\left(\frac{a+x}{2}\right)+f\left(\frac{b+x}{2}\right)-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right|$$
$$\leq(x-a)\int_0^{\frac{1}{2}}\lambda[|f'(\lambda x+(1-\lambda)a)|+|f'(\lambda a+(1-\lambda)x)|]d\lambda$$
$$+(b-x)\int_{\frac{1}{2}}^{1}(1-\lambda)[|f'(\lambda x+(1-\lambda)b)|+|f'(\lambda b+(1-\lambda)x)|]d\lambda$$
$$\leq(x-a)\int_0^{\frac{1}{2}}\lambda[|f'(a)|+\lambda\eta(|f'(x)|,|f'(a)|)+|f'(x)|+\lambda\eta(|f'(a)|,|f'(x)|)]d\lambda$$
$$+(b-x)\int_{\frac{1}{2}}^{1}(1-\lambda)[|f'(b)|+\lambda\eta(|f'(x)|,|f'(b)|)+|f'(x)|+\lambda\eta(|f'(b)|,|f'(x)|)]d\lambda$$
$$=\frac{(x-a)(|f'(a)|+|f'(x)|)+(b-x)(|f'(b)|+|f'(x)|)}{8}$$
$$+\frac{(x-a)(\eta(|f'(x)|,|f'(a)|))+(b-x)(\eta(|f'(b)|,|f'(x)|))}{12}$$

which completes the proof of (4.1).

Theorem 6 *Let* $f:[a,b]\subset[0,\infty)\to\mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $a<b$. *If* $|f'|^q$ *is* η*-convex for some* $q>1$, *then*

$$\left|f\left(\frac{a+x}{2}\right)+f\left(\frac{b+x}{2}\right)-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right| \tag{4.2}$$

$$\leq\frac{x-a}{(1+p)^{\frac{1}{p}}2^{1+\frac{1}{p}}}\left[\left(\frac{|f'(a)|^q}{2}+\frac{\eta(|f'(x)|^q,|f'(a)|^q)}{4}\right)^{\frac{1}{q}}+\left(\frac{|f'(x)|^q}{2}+\frac{\eta(|f'(a)|^q,|f'(x)|^q)}{4}\right)^{\frac{1}{q}}\right]$$

$$+\frac{b-x}{(1+p)^{\frac{1}{p}}2^{1+\frac{1}{p}}}\left[\left(\frac{|f'(b)|^q}{2}+\frac{\eta(|f'(x)|^q,|f'(b)|^q)}{4}\right)^{\frac{1}{q}}+\left(\frac{|f'(x)|^q}{2}+\frac{\eta(|f'(b)|^q,|f'(x)|^q)}{4}\right)^{\frac{1}{q}}\right]$$

where $x\in[a,b]$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2, by using Hölder inequality and $|f'|^q$ is η-convex, we have

$$\begin{aligned}
&\left|f\left(\frac{a+x}{2}\right)+f\left(\frac{b+x}{2}\right)-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right| \\
&\leq (x-a)\left(\int_0^{\frac{1}{2}}\lambda^p d\lambda\right)^{\frac{1}{p}}\left[\left(\int_0^{\frac{1}{2}}|f'(\lambda x+(1-\lambda)a)|^q\right)^{\frac{1}{q}}+\left(\int_0^{\frac{1}{2}}|f'(\lambda a+(1-\lambda)x)|^q\right)^{\frac{1}{q}}\right] \\
&+(b-x)\left(\int_{\frac{1}{2}}^1\lambda^p d\lambda\right)^{\frac{1}{p}}\left[\left(\int_{\frac{1}{2}}^1|f'(\lambda x+(1-\lambda)b)|^q\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1|f'(\lambda b+(1-\lambda)x)|^q\right)^{\frac{1}{q}}\right] \\
&\leq \frac{x-a}{(1+p)^{\frac{1}{p}}2^{1+\frac{1}{p}}}\left[\left(\int_0^{\frac{1}{2}}[|f'(a)|^q+\lambda\eta(|f'(x)|^q,|f'(a)|^q)]d\lambda\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^{\frac{1}{2}}[|f'(x)|^q+\lambda\eta(|f'(x)|^q,|f'(a)|^q)]d\lambda\right)^{\frac{1}{q}}\right] \\
&+\frac{b-x}{(1+p)^{\frac{1}{p}}2^{1+\frac{1}{p}}}\left[\left(\int_{\frac{1}{2}}^1[|f'(b)|^q+\lambda\eta(|f'(x)|^q,|f'(b)|^q)]d\lambda\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_{\frac{1}{2}}^1[|f'(x)|^q+\lambda\eta(|f'(b)|^q,|f'(x)|^q]d\lambda\right)^{\frac{1}{q}}\right] \\
&=\frac{x-a}{(1+p)^{\frac{1}{p}}2^{1+\frac{1}{p}}}\left[\left(\frac{|f'(a)|^q}{2}+\frac{\eta(|f'(x)|^q,|f'(a)|^q)}{4}\right)^{\frac{1}{q}}+\left(\frac{|f'(x)|^q}{2}+\frac{\eta(|f'(a)|^q,|f'(x)|^q)}{4}\right)^{\frac{1}{q}}\right] \\
&+\frac{b-x}{(1+p)^{\frac{1}{p}}2^{1+\frac{1}{p}}}\left[\left(\frac{|f'(b)|^q}{2}+\frac{\eta(|f'(x)|^q,|f'(b)|^q)}{4}\right)^{\frac{1}{q}}+\left(\frac{|f'(x)|^q}{2}+\frac{\eta(|f'(b)|^q,|f'(x)|^q)}{4}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

So, the proof is completed.

SIMPSON TYPE INEQUALITIES

Lemma 3 *[15] Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. $f'\in L[a,b]$, then the following equality holds:*

$$\frac{1}{3}\left[2f\left(\frac{a+x}{2}\right)+2f\left(\frac{b+x}{2}\right)+f(x)+\frac{f(a)+f(b)}{2}\right]$$

$$-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]$$
$$=(x-a)\int_0^{\frac{1}{2}}\left(\lambda-\frac{1}{6}\right)[f'(\lambda x+(1-\lambda)a)-f'(\lambda a+(1-\lambda)x)]d\lambda$$
$$+(b-x)\int_{\frac{1}{2}}^{1}\left(\frac{5}{6}-\lambda\right)[f'(\lambda x+(1-\lambda)b)-f'(\lambda b+(1-\lambda)x)]d\lambda$$

for $x\in[a,b]$.

Theorem 7 *Let* $f:[a,b]\subset[0,\infty)\to\mathrm{R}$ *be a differentiable mapping on* (a,b) *with* $a<b$. *If* $|f'|$ *is* η*-convex, then*

$$|\frac{1}{3}\left[2f\left(\frac{a+x}{2}\right)+2f\left(\frac{b+x}{2}\right)+f(x)+\frac{f(a)+f(b)}{2}\right] \tag{5.1}$$
$$-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]|$$
$$\leq\frac{5(x-a)[|f'(a)|+|f'(x)|]+5(b-x)[|f'(b)|,|f'(x)|]}{36}$$
$$+\frac{29(x-a)\eta(|f'(x)|,|f'(a)|)+61(b-x)\eta(|f'(b)|,|f'(x)|)}{648}$$

for $x\in[a,b]$.

Proof. From Lemma 3, by using the properties of modulus and $|f'|$ is η-convex, we have

$$|\frac{1}{3}\left[2f\left(\frac{a+x}{2}\right)+2f\left(\frac{b+x}{2}\right)+f(x)+\frac{f(a)+f(b)}{2}\right]$$
$$-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]|$$
$$\leq(x-a)\int_0^{\frac{1}{2}}\left|\lambda-\frac{1}{6}\right|[|f'(\lambda x+(1-\lambda)a)|+|f'(\lambda a+(1-\lambda)x)|]d\lambda$$
$$+(b-x)\int_{\frac{1}{2}}^{1}\left|\frac{5}{6}-\lambda\right|[|f'(\lambda x+(1-\lambda)b)|+|f'(\lambda b+(1-\lambda)x)|]d\lambda$$
$$\leq(x-a)\int_0^{\frac{1}{2}}\left|\lambda-\frac{1}{6}\right|[|f'(a)|+\lambda\eta(|f'(x)|,|f'(a)|)+|f'(x)|+\lambda\eta(|f'(a)|,|f'(x)|]d\lambda$$
$$+(b-x)\int_{\frac{1}{2}}^{1}\left|\frac{5}{6}-\lambda\right|[|f'(b)|+\lambda\eta(|f'(x)|,|f'(b)|)+|f'(x)|+\lambda\eta(|f'(b)|,|f'(x)|]d\lambda$$
$$=\frac{5(x-a)[|f'(a)|+|f'(x)|]+5(b-x)[|f'(b)|,|f'(x)|]}{36}$$
$$+\frac{29(x-a)\eta(|f'(x)|,|f'(a)|)+61(b-x)\eta(|f'(b)|,|f'(x)|)}{648}$$

which completes the proof of the inequality (5.1).

Theorem 8 *Let* $f:[a,b]\subset[0,\infty)\to\mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $a<b$. *If* $|f'|^q$ *is* η*-convex for some* $q>1$, *then*

$$
\begin{aligned}
&\left|\frac{1}{3}\left[2f\left(\frac{a+x}{2}\right)+2f\left(\frac{b+x}{2}\right)+f(x)+\frac{f(a)+f(b)}{2}\right]\right. \\
&\left.-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right| \\
&\le \frac{x-a}{(1+p)^{\frac{1}{p}}}\left[\frac{1}{3^{p+1}}+\frac{1}{6^{p+1}}\right]^{\frac{1}{p}} \\
&\times\left[\left(\frac{|f'(a)|^q}{2}+\frac{\eta(|f'(x)|^q,|f'(a)|^q)}{8}\right)^{\frac{1}{q}}+\left(\frac{f'(x)|^q}{2}+\frac{\eta(|f'(a)|^q,|f'(x)|^q)}{8}|\right)^{\frac{1}{q}}\right] \\
&+\frac{b-x}{(1+p)^{\frac{1}{p}}}\left[\frac{1}{3^{p+1}}+\frac{1}{6^{p+1}}\right]^{\frac{1}{p}} \\
&\times\left[\left(\frac{|f'(b)|^q}{2}+\frac{\eta(|f'(x)|^q,|f'(b)|^q)}{8}\right)^{\frac{1}{q}}+\left(\frac{|f'(x)|^q}{2}\right)^{\frac{1}{q}}+\frac{\eta(|f'(b)|^q,|f'(x)|^q)}{8}\right]
\end{aligned}
\tag{5.2}
$$

where $x\in[a,b]$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 3, by using Hölder inequality and $|f'|^q$ is η-convex, we have

$$
\left|\frac{1}{3}\left[2f\left(\frac{a+x}{2}\right)+2f\left(\frac{b+x}{2}\right)+f(x)+\frac{f(a)+f(b)}{2}\right]-\left[\frac{1}{x-a}\int_a^x f(t)dt+\frac{1}{b-x}\int_x^b f(t)dt\right]\right|
$$

$$
\le(x-a)\left(\int_0^{\frac{1}{2}}\left|\lambda-\frac{1}{6}\right|^p d\lambda\right)^{\frac{1}{p}}\left[\left(\int_0^{\frac{1}{2}}|f'(\lambda x+(1-\lambda)a)|^q\right)^{\frac{1}{q}}+\left(\int_0^{\frac{1}{2}}|f'(\lambda a+(1-\lambda)x)|^q\right)^{\frac{1}{q}}\right]
$$

$$
+(b-x)\left(\int_{\frac{1}{2}}^1\left|\frac{5}{6}-\lambda\right|^p d\lambda\right)^{\frac{1}{p}}\left[\left(\int_{\frac{1}{2}}^1|f'(\lambda x+(1-\lambda)b)|^q\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1|f'(\lambda b+(1-\lambda)x)|^q\right)^{\frac{1}{q}}\right]
$$

$$
\le\frac{1}{(1+p)^{\frac{1}{p}}}\left[\frac{1}{3^{p+1}}+\frac{1}{6^{p+1}}\right]^{\frac{1}{p}}
$$

$$\times\left\{(x-a)\left[\left(\int_0^{\frac{1}{2}}[|f'(a)|^q+\lambda\eta(|f'(x)|^q,|f'(a)|^q)]d\lambda\right)^{\frac{1}{q}}+\left(\int_0^{\frac{1}{2}}[|f'(x)|^q+\lambda\eta(|f'(a)|^q,|f'(x)|^q)]d\lambda\right)^{\frac{1}{q}}\right]\right.$$

$$\left.+(b-x)\left[\left(\int_{\frac{1}{2}}^{1}[|f'(b)|^q+\lambda\eta(|f'(x)|^q,|f'(b)|^q)]d\lambda\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^{1}[|f'(x)|^q+\lambda\eta(|f'(b)|^q,|f'(x)|^q)]d\lambda\right)^{\frac{1}{q}}\right]\right\}$$

$$=\frac{x-a}{(1+p)^{\frac{1}{p}}}\left[\frac{1}{3^{p+1}}+\frac{1}{6^{p+1}}\right]^{\frac{1}{p}}\left[\left(\frac{|f'(a)|^q}{2}+\frac{\eta(|f'(x)|^q,|f'(a)|^q)}{8}\right)^{\frac{1}{q}}+\left(\frac{f'(x)|^q}{2}+\frac{\eta(|f'(a)|^q,|f'(x)|^q)}{8}|\right)^{\frac{1}{q}}\right]$$

$$+\frac{b-x}{(1+p)^{\frac{1}{p}}}\left[\frac{1}{3^{p+1}}+\frac{1}{6^{p+1}}\right]^{\frac{1}{p}}\left[\left(\frac{|f'(b)|^q}{2}+\frac{\eta(|f'(x)|^q,|f'(b)|^q)}{8}\right)^{\frac{1}{q}}+\left(\frac{|f'(x)|^q}{2}\right)^{\frac{1}{q}}+\frac{\eta(|f'(b)|^q,|f'(x)|^q)}{8}\right].$$

So, the proof is completed.

REFERENCES

1. M.A. Ali, H. Kara, J. Tariboon, S. Asawasamrit, H. Budak, F. Hezenci, Some new Simpson's formula type inequalities for twice differentiable convex functions via generalized fractional operators, Symmetry, 13(12) (2021), Art. 2249.
2. P.S. Bullen, Error estimates for some elementary quadrature rules, Publikacije Elektrotehnickog fakulteta. Serija Matematika i fizika (602/633) (1978), 97-103.
3. M.R. Delavar, S.S. Dragomir, On η −convexity, Mathematical Inequalities and Applications, 20(1) (2017), 203-216.
4. T. Du, Y. Li, Z. Yang, A generalization of Simpson's inequality via differentiable mapping using extended (s, m)-convex functions, Appl. Math. Comput., 293 (2017), 358-369.
5. T. Du, C. Luo, Z. Cao, On the Bullen-type inequalities via generalized fractional integrals and their applications, Fractals 29(7) (2021), Article ID 2150188, 20 pages.
6. S. Erden, M.Z. Sarikaya, Generalized Bullen-type inequalities for local fractional integrals and its applications, Palest. J. Math., 9(2) (2020), 945-956.
7. F. Ertugral, M.Z. Sarikaya, Simpson type integral inequalities for generalized fractional integral, Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas, 113(4), 3115-3124 (2019).
8. L. Fejer, Uber die fourierreihen, ii, Math. Naturwise. Anz Ungar. Akad., Wiss, 24 (1906), 369-390.
9. M.E. Gordji, M.R. Delavar, S.S. Dragomir, Some inequality related to η-convex function, RGMIA 18 (2015), Art. 8., 1-14.
10. M.E. Gordji, S.S. Dragomir, M.R. Delavar, An inequality related to η-convex functions (II), Int. J. Nonlinear Anal. Appl., 6(2) (2015), 27-33.

11. M.E. Gordji, M.R. Delavar, M. De La Sen, On φ-convex functions, Journal of Mathematical Inequalities, 10(1) (2016), 173-183.
12. J. Hadamard, Etude sur les proprietes des fonctions entiÃ©res en particulier d'une fonction consideree par Riemann, J. Math. Pures Appl., 58 (1893), 171-215.
13. F. Hezenci, H. Budak, H. Kara, New version of Fractional Simpson type inequalities for twice differentiable functions, Adv. Differ. Equ. 2021(460), 1-10 (2021).
14. İ. İşcan, Hermite-Hadamard and Simpson-like type inequalities for differentiable harmonically convex functions, J. Math. (2014) Article ID 346305, 10 pages.
15. M.Z. Sarıkaya, On the some generalization of inequalities associated with Bullen, Simpson, Midpoint and Trapezoid type, ResearchGate, https://www.researchgate.net/publication/358884784.
16. M.Z. Sarıkaya, H. Budak, Some integral inequalities for local fractional integrals, Int. J. Anal. Appl., 14(1) (2017), 9-19.
17. E. Set, New inequalities of Ostrowski type for mappings whose derivatives are s-convex in the second sense via fractional integrals, Computers & Mathematics with Applications, 63(7) (2012), 1147-1154.
18. E. Set, A.O. Akdemir, M.E. Özdemir, Simpson type integral inequalities for convex functions via Riemann-Liouville integrals, Filomat, 31(14) (2017), 4415-4420.

New Estimations for P-Functions with the Help of Caputo-Fabrizio Fractional Integral Operators

Sinan ASLAN[1], Ahmet Ocak AKDEMİR[2] and Erhan SET[3]

[1]Ağrı Türk Telekom Social Sciences High School, Ağrı Türkiye
[2]Department of Mathematics, Faculty of Science and Arts,
Ağrı İbrahim Çeçen University, Ağrı-TURKEY
[23]Department of Mathematics, Faculty of Science and Arts,
Ordu University, Ordu-TURKEY

sinanaslan0407@gmail.com

ABSTRACT

In this paper, some novel integral inequalities for different kinds of convex functions have been proved by using Caputo-Fabrizio fractional integral operators. The findings includes several new integral inequalities P-functions. We have used the properties of Caputo-Fabrizio fractional operator, definitions of different kinds of convex functions and elemantery analysis methods.

INTRODUCTION

The concept of convexity, which has an important place in inequality theory, has been used by many researchers and has been used extensively, especially in the field of inequality theory. The definition of the convex functions can be given as follow.

Definition 1 *(See [1])* Let I be on interval in $\mathbb{R}$. Then $\varphi : I \to \mathbb{R}$ is said to be convex, if

$$\varphi(\zeta x + (1-\zeta)y) \leq \zeta\varphi(x) + (1-\zeta)\varphi(y)$$

holds for all $x, y \in I$ and $\zeta \in [0,1]$.
The definition of the P-functions can be given as follow.

Definition 2 *(See [2]) Let I be on interval in $\mathbb{R}$. Then $\varphi : I \to \mathbb{R}$ is $P-$ function or f belongs to the class of $P(I)$, if it is nonnegative and for all $x, y \in I$ and $\zeta \in [0,1]$, satisfies the following inequality*

$$\varphi(\zeta x + (1-\zeta)y) \leq \varphi(x) + \varphi(y)$$

For more information on the $P-$function, we recommend readers the following articles (see[3]-[9]).

Definition 3 *(See [10], [11], [12]) Let $\varphi \in H^1(a,b)$, $a < b$, $\alpha \in [0,1]$, then the definition of the left fractional derivative in the sense of Caputo and Fabrizio becomes*

$$\left({}_{a}^{CFC}D^{\alpha}\varphi\right)(\zeta)=\frac{B(\alpha)}{1-\alpha}\int_{a}^{t}\varphi'(x)e^{\frac{-\alpha(t-x)^{\alpha}}{1-\alpha}}dx$$

and the associated fractional integral is

$$\left({}_{a}^{CF}I^{\alpha}\varphi\right)(\zeta)=\frac{1-\alpha}{B(\alpha)}\varphi(\zeta)+\frac{\alpha}{B(\alpha)}\int_{a}^{t}\varphi(x)dx$$

where $B(\alpha)>0$ is a normalization function satisfying $B(0)=B(1)=1$. For the right fractional derivative we have

$$\left({}^{CFC}D_{b}^{\alpha}\varphi\right)(\zeta)=\frac{-B(\alpha)}{1-\alpha}\int_{t}^{b}\varphi'(x)e^{\frac{-\alpha(x-t)^{\alpha}}{1-\alpha}}dx$$

and the associated fractional integral is

$$\left({}^{CF}I_{b}^{\alpha}\varphi\right)(\zeta)=\frac{1-\alpha}{B(\alpha)}\varphi(\zeta)+\frac{\alpha}{B(\alpha)}\int_{t}^{b}\varphi(x)dx.$$

For more information related to different kinds of fractional operators, we recommend to the readers the following papers (See [13]-[28])

2 NEW INEQUALITIES FOR P-FUNCTIONS

Theorem 1 *Let $I\subseteq R$. Suppose that $f:[a,b]\subseteq I\to R$ is a $P-$function on $[a,b]$ such that $f\in L_1[a,b]$. Then, we have following inequality for Caputo-Fabrizio fractional integrals:*

$$\left({}_{a}^{CF}I^{\alpha}f\right)(k)+\left({}^{CF}I_{b}^{\alpha}f\right)(k)\leq\frac{2(1-\alpha)f(k)+\alpha(b-a)(f(a)+f(b))}{B(\alpha)}$$

where $B(\alpha)>0$ is normalization function $\alpha\in[0,1]$.

Proof. By using the definition of $P-$function, we can write

$$f(ta+(1-t)b)\leq f(a)+f(b).$$

By integrating both sides of the inequality over $[0,1]$ with respect to t, we get

$$\int_{0}^{1}f(ta+(1-t)b)dt\leq\int_{0}^{1}f(a)+f(b)dt.$$

By changing of the variable as $x=ta+(1-t)b$ and calculating the right hand side, we obtain

$$\frac{1}{b-a}\int_{a}^{b}f(x)dx\leq f(a)+f(b)$$

By multiplying both sides of the above inequality with $\frac{\alpha(b-a)}{B(\alpha)}$ and adding $\frac{2(1-\alpha)}{B(\alpha)}f(k)$, we have

$$\frac{2(1-\alpha)}{B(\alpha)}f(k)+\frac{\alpha}{B(\alpha)}\int_a^b f(x)dx\le\frac{2(1-\alpha)}{B(\alpha)}f(k)+\frac{\alpha(b-a)(f(a)+f(b))}{B(\alpha)}.$$

By simplfying the inequality, we get the result.

$$\left(\frac{(1-\alpha)}{B(\alpha)}f(k)+\frac{\alpha}{B(\alpha)}\int_a^k f(x)dx\right)+\left(\frac{(1-\alpha)}{B(\alpha)}f(k)+\frac{\alpha}{B(\alpha)}\int_k^b f(x)dx\right)$$
$$\le\frac{2(1-\alpha)}{B(\alpha)}f(k)+\frac{\alpha(b-a)(f(a)+f(b))}{B(\alpha)}.$$

Namely

$$\left({}_a^{CF}I^\alpha f\right)(k)+\left({}^{CF}I_b^\alpha f\right)(k)\le\frac{2(1-\alpha)f(k)+\alpha(b-a)(f(a)+f(b))}{B(\alpha)}.$$

Theorem 2 *Let $I\subseteq R$. Suppose that $f:[a,b]\subseteq I\to R$ is a $P-$function on $[a,b]$ such that $f\in L_1[a,b]$. Then, we have following inequality for Caputo-Fabrizio fractional integrals:*

$$\left({}_a^{CF}I^\alpha f\right)(k)+\left({}^{CF}I_b^\alpha f\right)(k)\le\frac{2(1-\alpha)}{B(\alpha)}f(k)+\frac{\alpha(b-a)}{B(\alpha)}\left(\frac{|f(a)|^p+|f(b)|^p}{p}+\frac{2}{q}\right)$$

where $B(\alpha)>0$ is normalization function $q>1,\frac{1}{p}+\frac{1}{q}=1$ and $\alpha\in[0,1]$.

Proof. By using the definition of $P-$function, we can write

$$f(ta+(1-t)b)\le f(a)+f(b).$$

By integrating both sides of the inequality over $[0,1]$ with respect to t, we get

$$\int_0^1|f(ta+(1-t)b)|dt\le\int_0^1|f(a)|+|f(b)|dt.$$

If we apply the Young's inequality to the right-hand side of the inequality, we get

$$\int_0^1|f(ta+(1-t)b)|dt\le\left(\frac{1}{p}\int_0^1|f(a)|^p dt+\frac{1}{q}\int_0^1 1^q dt\right)+\left(\frac{1}{p}\int_0^1|f(b)|^p dt+\frac{1}{q}\int_0^1 1^q dt\right).$$

By changing of the variable as $x=ta+(1-t)b$ and calculating the right hand side,we obtain

$$\frac{1}{b-a}\int_a^b f(x)dx \le \frac{|f(a)|^p+|f(b)|^p}{p}+\frac{2}{q}.$$

By multiplying both sides of the above inequality with $\frac{\alpha(b-a)}{B(\alpha)}$ and adding $\frac{2(1-\alpha)}{B(\alpha)}f(k)$, we have

$$\frac{2(1-\alpha)}{B(\alpha)}f(k)+\frac{\alpha}{B(\alpha)}\int_a^b f(x)dx \le \frac{2(1-\alpha)}{B(\alpha)}f(k)+\frac{\alpha(b-a)}{B(\alpha)}\left(\frac{|f(a)|^p+|f(b)|^p}{p}+\frac{2}{q}\right).$$

By simplfying the inequality, we get the result

$$\begin{aligned}&\left(\frac{(1-\alpha)}{B(\alpha)}f(k)+\frac{\alpha}{B(\alpha)}\int_a^k f(x)dx\right)+\left(\frac{(1-\alpha)}{B(\alpha)}f(k)+\frac{\alpha}{B(\alpha)}\int_k^b f(x)dx\right)\\&\le \frac{2(1-\alpha)}{B(\alpha)}f(k)+\frac{\alpha(b-a)}{B(\alpha)}\left(\frac{|f(a)|^p+|f(b)|^p}{p}+\frac{2}{q}\right).\end{aligned}$$

Namely

$$\left({}_a^{CF}I^{\alpha}f\right)(k)+\left({}^{CF}I_b^{\alpha}f\right)(k)\le \frac{2(1-\alpha)}{B(\alpha)}f(k)+\frac{\alpha(b-a)}{B(\alpha)}\left(\frac{|f(a)|^p+|f(b)|^p}{p}+\frac{2}{q}\right).$$

REFERENCES

[1] Pejaric, J.E., Tong, Y.L. (1992). Convex functions, partial orderings, and statistical applications. Academic Press.

[2] Dragomir, S.S., Pecaric, J., Persson, L.E. (1995). Some inequalities of Hadamard type. Soochow Journal of Mathematics, 21(3), 335-341.

[3] Akdemir, A.O., Aslan, S., Ekinci, A. (2022, October). Some New Inequalities for Exponentially $P-$ Functions on the Coordinates. In 5th International Conference on Mathematical and Related Sciences Book of Proceedings (p. 94-109).

[4] Pearce, C.E., Rubinov, A.M. (1999). P-functions, quasi-convex functions, and Hadamard-type inequalities. Journal of Mathematical Analysis and Applications, 240(1), 92-104.

[5] Set, E., Avci Ardic, M. (2017). Inequalities for log-convex functions and P-functions. Miskolc Mathematical Notes, 18(2), 1033-1041.

[6] İşcan, I., Olucak, V. (2019). Multiplicatively harmonically P-functions and some related inequalities. Sigma Journal of Engineering and Natural Sciences, 37(2), 521-528.

[7] Kadakal, H. (2020). On generalization of some integral inequalities for multiplicatively p-functions.

[8] İşcan, İ., Set, E., Özdemir, M.E. (2014). Some new general integral inequalities for P-functions. Malaya Journal of Matematik, 2(04), 510-516.

[9] Akdemir, A.O., Özdemir, M.E. (2010, November). Some Hadamard-Type Inequalities for Coordinated P-Convex Functions and Godunova-Levin Functions. In AIP Conference Proceedings (Vol. 1309, No. 1, pp. 7-15). American Institute of Physics.

[10] Abdeljawad, T, Baleanu, D. (2017). On fractional derivatives with exponential kernel and their discrete versions. J. Rep. Math. Phys. 80(1), 11-27 (2017).

[11] Abdeljawad, T. (2017). Fractional operators with exponential kernels and a Lyapunov type inequality. Advances in Difference Equations, 2017(1), 313.

[12] Caputo, M, Fabrizio, M. (2015). A new definition of fractional derivative without singular kernal. Prog. Fract. Differ. Appl. 1(2), 73-85 (2015).

[13] Abdeljawad, T. (2015). On conformable fractional calculus. Journal of Computational and Applied Mathematics, 279, 57-66.

[14] Abdeljawad, T., Baleanu, D. (2016). Integration by parts and its applications of a new nonlocal fractional derivative with Mittag-Leffler nonsingular kernel. arXiv preprint arXiv:1607.00262.

[15] Akdemir, A.O., Aslan, S., Çakaloğlu, M.N. Ekinci, A. (2021). Some New Results for Different Kinds of Convex Functions CaputoFabrizio Fractional Operators. 4th International Conference on Mathematical and Related Sciences. Page (92) ICMRS 2021.

[16] Akdemir, A. O., Aslan, S., Çakaloğlu, M.N., Set, E. (2021). New Hadamard Type Integral Inequalities via Caputo-Fabrizio Fractional Operators. 4th International Conference on Mathematical and Related Sciences. Page (91) ICMRS 2021.

[17] Aslan, S. (2023). Some Novel Fractional Integral Inequalities for Different Kinds of Convex Functions. Eastern Anatolian Journal of Science, 9 (1), 27-32.

[18] Akdemir, A., Aslan, S., Ekinci, A. (2022). Novel Approaches for s-Convex Functions via Caputo-Fabrizio Fractional Integrals. Proceedings of IAM, 11(1), 3-16.

[19] Akdemir, A. O., Butt, S. I., Nadeem, M., Ragusa, M. A. (2021). New general variants of Chebyshev type inequalities via generalized fractional integral operators. Mathematics, 9(2), 122.

[20] Akdemir, A. O., Ekinci, A., Set, E. (2017). Conformable fractional integrals and related

new integral inequalities.

[21] Butt, S.I., Umar, M., Rashid, S., Akdemir, A.O., Chu, Y.M. (2020). New Hermite-Jensen-Mercer-type inequalities via k-fractional integrals. Advances in Difference Equations, 2020, 1-24.

[22] Caputo, M., Fabrizio, M. (2015). A new definition of fractional derivative without singular kernel. Progress in Fractional Differentiation Applications, 1(2), 73-85.

[23] Gürbüz, M., Akdemir, A. O., Rashid, S., Set, E. (2020). Hermite-Hadamard inequality for fractional integrals of Caputo-Fabrizio type and related inequalities. Journal of Inequalities and Applications, 2020, 1-10.

[24] Ekinci, A., Özdemir, M. E. (2019). Some new integral inequalities via Riemann-Liouville integral operators. Applied and computational mathematics, 18(3).

[25] Rashid, S., Hammouch, Z., Kalsoom, H., Ashraf, R., Chu, Y. M. (2020). New investigation on the generalized K-fractional integral operators. Frontiers in Physics, 8, 25.

[26] Samko, S. G. (1993). Fractional integrals and derivatives. Theory and applications.

[27] Set, E., Akdemir, A. O., Özdemir, E.M. (2017). Simpson type integral inequalities for convex functions via Riemann-Liouville integrals. Filomat, 31(14), 4415-4420.

[28] Set, E. (2012). New inequalities of Ostrowski type for mappings whose derivatives are s-convex in the second sense via fractional integrals. Computers Mathematics with Applications, 63(7), 1147-1154.

Some Novel Integral Inequalities on the Co-ordinates for Log-Exponentially Convex Functions

Sinan ASLAN[1], Ahmet Ocak AKDEMİR[2] and Alper EKİNCİ[3]

[1]Ağrı Türk Telekom Social Sciences High School, Ağrı-TÜRKİYE
[2]Department of Mathematics, Faculty of Science and Arts,
Ağrı İbrahim Çeçen University, Ağrı-TURKEY
[2]Bandırma Vocational School,
Bandırma Onyedi Eylül University, Balikesir-TURKEY

sinanaslan0407@gmail.com

ABSTRACT

1 INTRODUCTION

The concept of convexity, which has an important place in inequality theory, has been widely used by many researchers, especially in the field of inequality theory. The definition of convex functions can be given in reference([2]) as follows.

Definition 1 *(See [2]) Let* I *be on interval in* $\mathbb{R}$. *Then* $\hat{o}: I \to \mathbb{R}$ *is said to be convex, if*

$$\hat{o}(\zeta\kappa_1 + (1-\zeta)\kappa_2) \le \zeta\hat{o}(\kappa_1) + (1-\zeta)\hat{o}(\kappa_2)$$

holds for all $\kappa_1, \kappa_2 \in I$ and $\zeta \in [0,1]$.

The main goal of studies on different types of convexity is to optimize the bounds and generalize some known classical inequalities. Based on this basic purpose, an important class of convex functions whose definition is given is exponential convex functions and whose definition ([1]) is given as follows.

Definition 2 *(See [1]) A function* $\hat{o}: I \subseteq \mathbb{R} \to \mathbb{R}$ *is said to be exponential convex function, if*

$$\hat{o}((1-\zeta)\kappa_1 + \zeta\kappa_2) \le (1-\zeta)\frac{\hat{o}(\kappa_1)}{e^{\alpha\kappa_2}} + \zeta\frac{\hat{o}(\kappa_2)}{e^{\alpha\kappa_2}}$$

for all $\kappa_1, \kappa_2 \in I, \alpha \in \mathbb{R}$ and $\zeta \in [0,1]$.

In [3], the concept of log-convex functions was introduced as follows.

Definition 3 *A function* $\hat{o}: I \to (0,\infty)$ *is said to be a log-convex function, if*

$$\hat{o}(\zeta\kappa_1 + (1-\zeta)\kappa_2) \le [\hat{o}(\kappa_1)]^{\zeta}[\hat{o}(\kappa_2)]^{1-\zeta}$$

for all $\kappa_1, \kappa_2 \in I$ and $\zeta \in [0,1]$.
For some recent results connected with log-convex functions, (See [4]-[10]).

Aslan and Akdemir gave the definition of exponentially convex functions on co-ordinates as follows:

Definition 4 *(See [11]) Let us consider the bidimensional interval* $\Delta = [\kappa_1, \kappa_2] \times \kappa_3, \kappa_4]$ *in* R^2 *with* $\kappa_1 < \kappa_2$ *and* $\kappa_3 < \kappa_4$. *The mapping* $\hat{o} : \Delta \to R$ *is exponential convex on the co-ordinates on* Δ, *if the following inequality holds,*

$$\hat{o}(\zeta\kappa_1 + (1-\zeta)\kappa_3, \zeta\kappa_2 + (1-\zeta)\kappa_4) \le \zeta \frac{\hat{o}(\kappa_1, \kappa_2)}{e^{\alpha(\kappa_1+\kappa_2)}} + (1-\zeta)\frac{\hat{o}(\kappa_3, \kappa_4)}{e^{\alpha(\kappa_3+\kappa_4)}}$$

for all $(\kappa_1, \kappa_2), (\kappa_3, \kappa_4) \in \Delta, \alpha \in R,$ and $\zeta \in [0,1]$.

Aslan and Akdedmir gave an equivalent definition of the exponential convex function definition on the coordinates as follows:

Definition 5 *(see [11]) The mapping* $\hat{o} : \Delta \to \mathrm{R}$ *is exponential convex on the co-ordinates on* Δ, *if the following inequality holds,*

$$\begin{aligned} &\hat{o}(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) \\ &\le \zeta\xi \frac{\hat{o}(\kappa_1, \kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} + \zeta(1-\xi)\frac{\hat{o}(\kappa_1, \kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} + (1-\zeta)\xi\frac{\hat{o}(\kappa_2, \kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} + (1-\zeta)(1-\xi)\frac{\hat{o}(\kappa_2, \kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}} \end{aligned}$$

for all $(\kappa_1, \kappa_3), (\kappa_1, \kappa_4), (\kappa_2, \kappa_3), (\kappa_2, \kappa_4) \in \Delta, \alpha \in \mathrm{R}$ and $\zeta, \xi \in [0,1]$.

Expressing convex functions in coordinates brought up the question that it is possible for Hermite-Hadamard inequality to expand into coordinates. The answer to this motivating question has been found in Dragomir's paper (see [12]) and has taken its place in the literature as the expansion of Hermite-Hadamard inequality to a rectangle from the plane R^2 stated below.

Theorem 1 *(see [12]) Suppose that* $\hat{o} : \Delta = [\kappa_1, \kappa_2] \times \kappa_3, \kappa_4] \to \mathrm{R}$ *is convex on the co-ordinates on* Δ. *Then one has the inequalities;*

$$\begin{aligned} &\hat{o}\left(\frac{\kappa_1+\kappa_2}{2}, \frac{\kappa_3+\kappa_4}{2}\right) \le \frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4} \hat{o}(x,y)\,dxdy \\ &\le \frac{\hat{o}(\kappa_1, \kappa_3) + \hat{o}(\kappa_1, \kappa_4) + \hat{o}(\kappa_2, \kappa_3) + \hat{o}(\kappa_2, \kappa_4)}{4}. \end{aligned}$$

The above inequalities are sharp.

Aslan and Akdemir extended Dragomir's result in Theorem 1 to exponantially convexity on the coordinates

Theorem 2 *(See [11]) Let* $\hat{o}:\Delta=[\kappa_1,\kappa_2]\times\kappa_3,\kappa_4]\to\mathrm{R}$ *be partial differentiable mapping on* $\Delta=[\kappa_1,\kappa_2]\times\kappa_3,\kappa_4]$ *and* $\hat{o}\in L(\Delta)$, $\alpha\in\mathrm{R}$. *If* $\hat{o}$ *is exponential convex function on the co-ordinates on* Δ, *then the following inequality holds;*

$$\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\hat{o}(x,y)dxdy\leq\frac{\frac{\hat{o}(\kappa_1,\kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}}+\frac{\hat{o}(\kappa_1,\kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}}+\frac{\hat{o}(\kappa_2,\kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}}+\frac{\hat{o}(\kappa_2,\kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}}}{4}.$$

For some recent results connected with exponentially convex functions and exponentially convex functions on the co-ordinates, (See [13]-[21]).

Anderson et al. gave the following definition in (See [22])

Definition 6 *A function* $M:(0,\infty)\times(0,\infty)\to(0,\infty)$ *is called a Mean function if*

(1) $M(\kappa_1,\kappa_2)=M(\kappa_2,\kappa_1)$,
(2) $M(\kappa_1,\kappa_1)=\kappa_1$,
(3) $\kappa_1<M(\kappa_1,\kappa_2)<\kappa_2$, whenever $\kappa_1<\kappa_2$,
(4) $M(a\kappa_1,a\kappa_2)=aM(\kappa_1,\kappa_2)$ for all $a>0$.

Let us recall special means (See [22], [23] and [24])

1. Arithmetic Mean: $M(\kappa_1,\kappa_2)=A(\kappa_1,\kappa_2)=\frac{\kappa_1+\kappa_2}{2}$.
2. Geometric Mean: $M(\kappa_1,\kappa_2)=G(\kappa_1,\kappa_2)=\sqrt{\kappa_1\kappa_2}$.
3. Harmonic Mean: $M(\kappa_1,\kappa_2)=H(\kappa_1,\kappa_2)=1/A\left(\frac{1}{\kappa_1},\frac{1}{\kappa_2}\right)$.
4. Logarithmic Mean: $M(\kappa_1,\kappa_2)=L(\kappa_1,\kappa_2)=(\kappa_1-\kappa_2)/(\log\kappa_1-\log\kappa_2)$ for $\kappa_1\neq\kappa_2$ and $L(\kappa_1,\kappa_1)=\kappa_1$.
5. Identric Mean: $M(\kappa_1,\kappa_2)=I(\kappa_1,\kappa_2)=(1/e)\left(\kappa_1^{\kappa_1}/\kappa_2^{\kappa_2}\right)^{1/(\kappa_1-\kappa_2)}$ for $\kappa_1\neq\kappa_2$ and $I(\kappa_1,\kappa_1)=\kappa_1$.

Now we are in a position to put in order as:

$$H(\kappa_1,\kappa_2)\leq G(\kappa_1,\kappa_2)\leq L(\kappa_1,\kappa_2)\leq I(\kappa_1,\kappa_2)\leq A(\kappa_1,\kappa_2)\leq K(\kappa_1,\kappa_2).$$

In [22], authors also gave a definition which is called $MN-$convexity as the following:

Definition 7 *Let* $\hat{o}: I \to (0,\infty)$ *be continuous, where* I *is subinterval of* $(0,\infty)$*. Let* M *and* N *be any two Mean functions. We say* $\hat{o}$ *is* MN *-convex (concave) if*

$$\hat{o}(M(\kappa_1,\kappa_2)) \leq (\geq) N(\hat{o}(\kappa_1), \hat{o}(\kappa_2))$$

for all $\kappa_1, \kappa_2 \in I$.

In this study, $AG-$convex functions on the co-ordinates have been introduced and a fundamental integral inequality of Hadamard-type has been proved for $AG-$exponentially convex functions on the co-ordinates.

2 MAIN RESULTS

Definition 8 *Let us consider the bidimensional interval* $\Delta = [\kappa_1,\kappa_2] \times \kappa_3,\kappa_4]$ *in* R^2 *with* $\kappa_1 < \kappa_2$ *and* $\kappa_3 < \kappa_4$*. The mapping* $\hat{o}: \Delta \to R^+$ *is* $AG-$*exponentially convex on the co-ordinates on* Δ *, if the following inequality holds,*

$$\hat{o}(\zeta\kappa_1 + (1-\zeta)\kappa_3, \zeta\kappa_2 + (1-\zeta)\kappa_4) \leq \frac{\hat{o}^{\zeta}(\kappa_1,\kappa_2)}{e^{\alpha(\kappa_1+\kappa_2)}} \frac{\hat{o}^{(1-\zeta)}(\kappa_3,\kappa_4)}{e^{\alpha(\kappa_3+\kappa_4)}}$$

for all $(\kappa_1,\kappa_2),(\kappa_3,\kappa_4) \in \Delta, \alpha \in R$ and $\zeta \in [0,1]$.

An equivalent definition of the $AG-$exponentially convex function definition on the coordinates can be done as follows:

Definition 9 *The mapping* $\hat{o}: \Delta \to R^+$ *is* $AG-$*exponential convex on the co-ordinates on* Δ*, if the following inequality holds,*

$$\begin{aligned}&\hat{o}(\zeta\kappa_1 + (1-\zeta)\kappa_2, \xi\kappa_3 + (1-\xi)\kappa_4) \\ &\leq \frac{\hat{o}^{\zeta\xi}(\kappa_1,\kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}} \frac{\hat{o}^{\zeta(1-\xi)}(\kappa_1,\kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}} \frac{\hat{o}^{(1-\zeta)\xi}(\kappa_2,\kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}} \frac{\hat{o}^{(1-\zeta)(1-\xi)}(\kappa_2,\kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}}\end{aligned}$$

for all $(\kappa_1,\kappa_3),(\kappa_1,\kappa_4),(\kappa_2,\kappa_3),(\kappa_2,\kappa_4) \in \Delta, \alpha \in R$ and $\zeta,\xi \in [0,1]$.

Lemma 1 *A function* $\hat{o}: \Delta \to R^+$ *will be called* $AG-$*exponential convex on the co-ordinates on* Δ*, if the partial mappings* $\hat{o}_{\tau_2}: [\kappa_1,\kappa_2] \to R$ *,* $\hat{o}_{\tau_2}(u) = e^{\alpha\tau_2} f(u,\tau_2)$ *and* $\hat{o}_{\tau_1}: [\kappa_3,\kappa_4] \to R$ *,* $\hat{o}_{\tau_1}(v) = e^{\alpha\tau_1} f(\tau_1, v)$ *are* $AG-$*exponential convex on the co-ordinates on* Δ*, where defined for all* $\tau_2 \in \kappa_3,\kappa_4]$ *and* $\tau_1 \in \kappa_1,\kappa_2]$*.*

Proof. From the definition of partial mapping $\hat{o}_{\tau_1}$ we can write

$$
\begin{aligned}
&\hat{o}_{\tau_1}(\zeta v_1+(1-\zeta)v_2)=e^{\alpha\tau_1}\hat{o}(\tau_1,\zeta v_1+(1-\zeta)v_2)\\
&=e^{\alpha\tau_1}\hat{o}(\zeta\tau_1+(1-\zeta)\tau_1,\zeta v_1+(1-\zeta)v_2)\\
&\le e^{\alpha\tau_1}\left[\zeta\frac{\hat{o}(\tau_1,v_1)}{e^{\alpha(\tau_1+v_1)}}+(1-\zeta)\frac{\hat{o}(\tau_1,v_2)}{e^{\alpha(\tau_1+v_2)}}\right]\\
&=\zeta\frac{\hat{o}(\tau_1,v_1)}{e^{\alpha v_1}}+(1-\zeta)\frac{\hat{o}(\tau_1,v_2)}{e^{\alpha v_2}}\\
&=\zeta\frac{\hat{o}_{\tau_1}(v_1)}{e^{\alpha v_1}}+(1-\zeta)\frac{\hat{o}_{\tau_1}(v_2)}{e^{\alpha v_2}}.
\end{aligned}
$$

Similarly, one can easily see that

$$
\hat{o}_{\tau_2}(\zeta u_1+(1-\zeta)u_2)\le\zeta\frac{\hat{o}_{\tau_2}(u_1)}{e^{\alpha u_1}}+(1-\zeta)\frac{\hat{o}_{\tau_2}(u_2)}{e^{\alpha u_2}}.
$$

The proof is completed.

Proposition 1 *If* $\hat{o},\Phi:\Delta\to R$ *are two* $AG-$*exponential convex functions on the co-ordinates on* Δ, *then* $\hat{o}\Phi$ *is* $AG-$*exponential convex functions on the co-ordinates on* Δ.

Proof. It is easy to see that

$$
\begin{aligned}
&\hat{o}(\zeta\tau_1+(1-\zeta)\tau_2,\xi\tau_3+(1-\xi)\tau_4)\\
&\times\Phi(\zeta\tau_1+(1-\zeta)\tau_2,\xi\tau_3+(1-\xi)\tau_4)\\
&\le\frac{\hat{o}^{\zeta\xi}(\kappa_1,\kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}}\frac{\hat{o}^{\zeta(1-\xi)}(\kappa_1,\kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}}\frac{\hat{o}^{(1-\zeta)\xi}(\kappa_2,\kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}}\frac{\hat{o}^{(1-\zeta)(1-\xi)}(\kappa_2,\kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}}\\
&\times\frac{\Phi^{\zeta\xi}(\kappa_1,\kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}}\frac{\Phi^{\zeta(1-\xi)}(\kappa_1,\kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}}\frac{\Phi^{(1-\zeta)\xi}(\kappa_2,\kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}}\frac{\Phi^{(1-\zeta)(1-\xi)}(\kappa_2,\kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}}\\
&=\frac{(\hat{o}\Phi)^{\zeta\xi}(\kappa_1,\kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}}\frac{(\hat{o}\Phi)^{\zeta(1-\xi)}(\kappa_1,\kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}}\frac{(\hat{o}\Phi)^{(1-\zeta)\xi}(\kappa_2,\kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}}\frac{(\hat{o}\Phi)^{(1-\zeta)(1-\xi)}(\kappa_2,\kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}}.
\end{aligned}
$$

Therefore $\hat{o}\Phi$ is $AG-$exponential convex functions on the co-ordinates on Δ.

Theorem 3 *Let* $\hat{o}:\Delta=[\kappa_1,\kappa_2]\times\kappa_3,\kappa_4]\to R^+$ *be partial differentiable mapping on* $\Delta=[\kappa_1,\kappa_2]\times\kappa_3,\kappa_4]$ *and* $\hat{o}\in L(\Delta)$, $\alpha\in R$. *If* $\hat{o}$ *is* $AG-$*exponential convex function on the co-ordinates on* Δ, *then the following inequality holds;*

$$
\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\hat{o}(\tau_1,\tau_2)d\tau_1d\tau_2
$$

$$\leq \frac{L(\hat{o}(\kappa_1,\kappa_3),\hat{o}(\kappa_1,\kappa_4))+L(\hat{o}(\kappa_2,\kappa_3)\hat{o}(\kappa_2,\kappa_4))}{2e^{2\alpha(\kappa_1+\kappa_3+\kappa_1+\kappa_4)}}.$$

where $\tau_1 \in [\kappa_1,\kappa_2]$ and $\tau_2 \in [\kappa_3,\kappa_4]$ dir.

Proof. By the definition of the $AG-$exponential convex functions on the co-ordinates on Δ, we can write

$$\begin{aligned}
&\hat{o}(\zeta\tau_1+(1-\zeta)\tau_2,\xi\tau_3+(1-\xi)\tau_4)\\
&\leq \frac{\hat{o}^{\zeta\xi}(\kappa_1,\kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}}\frac{\hat{o}^{\zeta(1-\xi)}(\kappa_1,\kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}}\frac{\hat{o}^{(1-\zeta)\xi}(\kappa_2,\kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}}\frac{\hat{o}^{(1-\zeta)(1-\xi)}(\kappa_2,\kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}}
\end{aligned}$$

By integrating both sides of the above inequality with respect to ζ,ξ on $[0,1]^2$, we have

$$\begin{aligned}
&\int_0^1\int_0^1 \hat{o}(\zeta\kappa_1+(1-\zeta)\kappa_2,\xi\kappa_3+(1-\xi)\kappa_4)\,d\zeta d\xi\\
&\leq \int_0^1\int_0^1 \frac{\hat{o}^{\zeta\xi}(\kappa_1,\kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}}\frac{\hat{o}^{\zeta(1-\xi)}(\kappa_1,\kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}}\frac{\hat{o}^{(1-\zeta)\xi}(\kappa_2,\kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}}\frac{\hat{o}^{(1-\zeta)(1-\xi)}(\kappa_2,\kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}}\,d\zeta d\xi\\
&= \frac{1}{e^{2\alpha(\kappa_1+\kappa_3+\kappa_1+\kappa_4)}}\int_0^1\int_0^1 \hat{o}^{\zeta\xi}(\kappa_1,\kappa_3)\hat{o}^{\zeta(1-\xi)}(\kappa_1,\kappa_4)\hat{o}^{(1-\zeta)\xi}(\kappa_2,\kappa_3)\hat{o}^{(1-\zeta)(1-\xi)}(\kappa_2,\kappa_4)\,d\zeta d\xi\\
&= \frac{1}{e^{2\alpha(\kappa_1+\kappa_3+\kappa_1+\kappa_4)}}\int_0^1 \frac{\hat{o}^{\xi}(\kappa_1,\kappa_3)\hat{o}^{(1-\xi)}(\kappa_1,\kappa_4)-\hat{o}^{\xi}(\kappa_2,\kappa_3)\hat{o}^{(1-\xi)}(\kappa_2,\kappa_4)}{\ln\hat{o}^{\xi}(\kappa_1,\kappa_3)\hat{o}^{(1-\xi)}(\kappa_1,\kappa_4)-\ln\hat{o}^{\xi}(\kappa_2,\kappa_3)\hat{o}^{(1-\xi)}(\kappa_2,\kappa_4)}\,d\xi\\
&= \frac{1}{e^{2\alpha(\kappa_1+\kappa_3+\kappa_1+\kappa_4)}}\int_0^1 L(\hat{o}^{\xi}(\kappa_1,\kappa_3)\hat{o}^{(1-\xi)}(\kappa_1,\kappa_4),\hat{o}^{\xi}(\kappa_2,\kappa_3)\hat{o}^{(1-\xi)}(\kappa_2,\kappa_4))\,d\xi
\end{aligned}$$

If the $\tau_1=\zeta\kappa_1+(1-\zeta)\kappa_2, \tau_2=\xi\kappa_3+(1-\xi)\kappa_4$ variable is changed and the $L(\kappa_1,\kappa_2)<A(\kappa_1,\kappa_2)$ feature is taken into account, the following result is obtained.

$$\begin{aligned}
&\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\hat{o}(\tau_1,\tau_2)\,d\tau_1 d\tau_2\\
&\leq \frac{1}{e^{2\alpha(\kappa_1+\kappa_3+\kappa_1+\kappa_4)}}\int_0^1 A(\hat{o}^{\xi}(\kappa_1,\kappa_3)\hat{o}^{(1-\xi)}(\kappa_1,\kappa_4),\hat{o}^{\xi}(\kappa_2,\kappa_3)\hat{o}^{(1-\xi)}(\kappa_2,\kappa_4))\,d\xi\\
&= \frac{1}{e^{2\alpha(\kappa_1+\kappa_3+\kappa_1+\kappa_4)}}\int_0^1 \frac{\hat{o}^{\xi}(\kappa_1,\kappa_3)\hat{o}^{(1-\xi)}(\kappa_1,\kappa_4)+\hat{o}^{\xi}(\kappa_2,\kappa_3)\hat{o}^{(1-\xi)}(\kappa_2,\kappa_4)}{2}\,d\xi\\
&= \frac{L(\hat{o}(\kappa_1,\kappa_3),\hat{o}(\kappa_1,\kappa_4))+L(\hat{o}(\kappa_2,\kappa_3)\hat{o}(\kappa_2,\kappa_4))}{2e^{2\alpha(\kappa_1+\kappa_3+\kappa_1+\kappa_4)}}.
\end{aligned}$$

By computing the above integrals, we obtain the desired result.

Corollary 1 *If we choose* $\alpha = 0$ *in Theorem 3, the result agrees* $AG-$ *exponential convex on the coordinates with* $AG-$ *convexity on the coordinates*

$$\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\hat{o}(\tau_1,\tau_2)d\tau_1 d\tau_2 \leq \frac{L(\hat{o}(\kappa_1,\kappa_3),\hat{o}(\kappa_1,\kappa_4))+L(\hat{o}(\kappa_2,\kappa_3)\hat{o}(\kappa_2,\kappa_4))}{2}.$$

where $\tau_1 \in [\kappa_1,\kappa_2]$ and $\tau_2 \in [\kappa_3,\kappa_4]$ dir.

Theorem 4 *Let* $\hat{o}:\Delta=[\kappa_1,\kappa_2]\times\kappa_3,\kappa_4]\to R_+$ *be partial differentiable mapping on* $\Delta=[\kappa_1,\kappa_2]\times\kappa_3,\kappa_4]$ *and* $\hat{o}\in L(\Delta)$, $\alpha\in R$. *If* $|\hat{o}|$ *is* $AG-$ *exponential convex function on the co-ordinates on* Δ, $p>1$ *then the following inequality holds;*

$$\left|\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\hat{o}(\tau_1,\tau_2)d\tau_1 d\tau_2\right| \leq \left(\frac{1}{e^{2p\alpha(\kappa_1+\kappa_3+\kappa_1+\kappa_4)}}\right)^{\frac{1}{p}} \times\left(\frac{L\left(|\hat{o}(\kappa_1,\kappa_3)|^q,|\hat{o}(\kappa_1,\kappa_4)|^q\right)+L\left(|\hat{o}(\kappa_2,\kappa_3)|^q,|\hat{o}(\kappa_2,\kappa_4)|^q\right)}{2}\right)^{\frac{1}{q}}$$

where $\tau_1\in[\kappa_1,\kappa_2]$, $\tau_2\in[\kappa_3,\kappa_4]$ and $p^{-1}+q^{-1}=1$ dir.

Proof. By the definition of the $AG-$ exponential convex functions on the co-ordinates on Δ, we can write

$$\hat{o}(\zeta\kappa_1+(1-\zeta)\kappa_2,\xi\kappa_3+(1-\xi)\kappa_4) \leq \frac{\hat{o}^{\zeta\xi}(\kappa_1,\kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}}\frac{\hat{o}^{\zeta(1-\xi)}(\kappa_1,\kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}}\frac{\hat{o}^{(1-\zeta)\xi}(\kappa_2,\kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}}\frac{\hat{o}^{(1-\zeta)(1-\xi)}(\kappa_2,\kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}}.$$

The absolute value property is used in integral and by integrating both sides of the above inequality with respect to ζ,ξ on $[0,1]^2$, we can write

$$\left|\int_0^1\int_0^1\hat{o}(\zeta\kappa_1+(1-\zeta)\kappa_2,\xi\kappa_3+(1-\xi)\kappa_4)d\zeta d\xi\right| \leq \int_0^1\int_0^1\left|\frac{\hat{o}^{\zeta\xi}(\kappa_1,\kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}}\frac{\hat{o}^{\zeta(1-\xi)}(\kappa_1,\kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}}\frac{\hat{o}^{(1-\zeta)\xi}(\kappa_2,\kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}}\frac{\hat{o}^{(1-\zeta)(1-\xi)}(\kappa_2,\kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}}\right|d\zeta d\xi$$

If the $\tau_1 = \zeta\kappa_1 + (1-\zeta)\kappa_2, \tau_2 = \xi\kappa_3 + (1-\xi)\kappa_4$ variable is changed and If we apply the Hölder's inequality to the right-hand side of the inequality, we get

$$\left|\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\hat{o}(\tau_1,\tau_2)d\tau_1 d\tau_2\right|$$

$$\leq\left(\int_0^1\int_0^1\frac{1}{e^{2p\alpha(\kappa_1+\kappa_3+\kappa_1+\kappa_4)}}d\zeta d\xi\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1\int_0^1|\hat{o}(\kappa_1,\kappa_3)|^{\zeta\xi q}|\hat{o}(\kappa_1,\kappa_4)|^{\zeta(1-\xi)q}|\hat{o}(\kappa_2,\kappa_3)|^{(1-\zeta)\xi q}|\hat{o}(\kappa_2,\kappa_4)|^{(1-\zeta)(1-\xi)q}d\zeta d\xi\right)^{\frac{1}{q}}$$

$$=\left(\frac{1}{e^{2p\alpha(\kappa_1+\kappa_3+\kappa_1+\kappa_4)}}\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 L\left(|\hat{o}(\kappa_1,\kappa_3)|^{\xi q}|\hat{o}(\kappa_1,\kappa_4)|^{(1-\xi)q},|\hat{o}(\kappa_2,\kappa_3)|^{\xi q}|\hat{o}(\kappa_2,\kappa_4)|^{(1-\xi)q}\right)d\xi\right)^{\frac{1}{q}}.$$

Because of the $L(\kappa_1,\kappa_2) < A(\kappa_1,\kappa_2)$ property, we can write

$$\left|\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\hat{o}(\tau_1,\tau_2)d\tau_1 d\tau_2\right|$$

$$\leq\left(\frac{1}{e^{2p\alpha(\kappa_1+\kappa_3+\kappa_1+\kappa_4)}}\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 A\left(|\hat{o}(\kappa_1,\kappa_3)|^{\xi q}|\hat{o}(\kappa_1,\kappa_4)|^{(1-\xi)q},|\hat{o}(\kappa_2,\kappa_3)|^{\xi q}|\hat{o}(\kappa_2,\kappa_4)|^{(1-\xi)q}\right)d\xi\right)^{\frac{1}{q}}$$

$$=\left(\frac{1}{e^{2p\alpha(\kappa_1+\kappa_3+\kappa_1+\kappa_4)}}\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1\frac{|\hat{o}(\kappa_1,\kappa_3)|^{\xi q}|\hat{o}(\kappa_1,\kappa_4)|^{(1-\xi)q}+|\hat{o}(\kappa_2,\kappa_3)|^{\xi q}|\hat{o}(\kappa_2,\kappa_4)|^{(1-\xi)q}}{2}d\xi\right)^{\frac{1}{q}}$$

$$=\left(\frac{1}{e^{2p\alpha(\kappa_1+\kappa_3+\kappa_1+\kappa_4)}}\right)^{\frac{1}{p}}$$

$$\times\left(\frac{L\left(|\hat{o}(\kappa_1,\kappa_3)|^{q}|\hat{o}(\kappa_1,\kappa_4)|^{q}\right)+L\left(|\hat{o}(\kappa_2,\kappa_3)|^{q}|\hat{o}(\kappa_2,\kappa_4)|^{q}\right)}{2}\right)^{\frac{1}{q}}.$$

Proof is completed.

Corollary 2 *If we choose* $\alpha = 0$ *in Theorem 4, the result agrees* $AG-$*exponential convex on the coordinates with log-convexity on the coordinates.*

$$\left|\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\hat{o}(\tau_1,\tau_2)d\tau_1 d\tau_2\right|$$
$$\leq\left(\frac{L\left(|\hat{o}(\kappa_1,\kappa_3)|^q|\hat{o}(\kappa_1,\kappa_4)|^q\right)+L\left(|\hat{o}(\kappa_2,\kappa_3)|^q|\hat{o}(\kappa_2,\kappa_4)|^q\right)}{2}\right)^{\frac{1}{q}}$$

where $\tau_1\in[\kappa_1,\kappa_2]$, $\tau_2\in[\kappa_3,\kappa_4]$ and $p^{-1}+q^{-1}=1$ dir.

Theorem 5 *Let* $\hat{o}:\Delta=[\kappa_1,\kappa_2]\times\kappa_3,\kappa_4]\to R^+$ *be partial differentiable mapping on* $\Delta=[\kappa_1,\kappa_2]\times\kappa_3,\kappa_4]$ *and* $\hat{o}\in L(\Delta)$, $\alpha\in R$. *If* $|\hat{o}|$ *is* $AG-$*exponential convex function on the co-ordinates on* Δ, $p>1$ *then the following inequality holds;*

$$\left|\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\hat{o}(\tau_1,\tau_2)d\tau_1 d\tau_2\right|$$
$$\leq\frac{1}{pe^{2p\alpha(\kappa_1+\kappa_3+\kappa_1+\kappa_4)}}$$
$$+\frac{L\left(|\hat{o}(\kappa_1,\kappa_3)|^q|\hat{o}(\kappa_1,\kappa_4)|^q\right)+L\left(|\hat{o}(\kappa_2,\kappa_3)|^q|\hat{o}(\kappa_2,\kappa_4)|^q\right)}{2q}$$

where $\tau_1\in[\kappa_1,\kappa_2]$, $\tau_2\in[\kappa_3,\kappa_4]$ and $p^{-1}+q^{-1}=1$ dir.

Proof. By the definition of the $AG-$exponential convex functions on the co-ordinates on Δ, we can write

$$\hat{o}(\zeta\tau_1+(1-\zeta)\tau_2,\xi\tau_3+(1-\xi)\tau_4)$$
$$\leq\frac{\hat{o}^{\zeta\xi}(\kappa_1,\kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}}\frac{\hat{o}^{\zeta(1-\xi)}(\kappa_1,\kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}}\frac{\hat{o}^{(1-\zeta)\xi}(\kappa_2,\kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}}\frac{\hat{o}^{(1-\zeta)(1-\xi)}(\kappa_2,\kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}}.$$

The absolute value property is used in integral and by integrating both sides of the above inequality with respect to ζ,ξ on $[0,1]^2$, we can write

$$\left|\int_0^1\int_0^1\hat{o}(\zeta\tau_1+(1-\zeta)\tau_2,\xi\tau_3+(1-\xi)\tau_4)d\zeta d\xi\right|$$
$$\leq\int_0^1\int_0^1\left|\frac{\hat{o}^{\zeta\xi}(\kappa_1,\kappa_3)}{e^{\alpha(\kappa_1+\kappa_3)}}\frac{\hat{o}^{\zeta(1-\xi)}(\kappa_1,\kappa_4)}{e^{\alpha(\kappa_1+\kappa_4)}}\frac{\hat{o}^{(1-\zeta)\xi}(\kappa_2,\kappa_3)}{e^{\alpha(\kappa_2+\kappa_3)}}\frac{\hat{o}^{(1-\zeta)(1-\xi)}(\kappa_2,\kappa_4)}{e^{\alpha(\kappa_2+\kappa_4)}}\right|d\zeta d\xi.$$

If the $\tau_1 = \zeta\kappa_1 + (1-\zeta)\kappa_2, \tau_2 = \xi\kappa_3 + (1-\xi)\kappa_4$ variable is changed and If we apply the Young's inequality to the right-hand side of the inequality, we get

$$\begin{aligned}
&\left|\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\hat{o}(\tau_1,\tau_2)d\tau_1 d\tau_2\right| \\
&\leq \frac{1}{p}\left(\int_0^1\int_0^1 \frac{1}{e^{2p\alpha(\kappa_1+\kappa_3+\kappa_1+\kappa_4)}}d\zeta d\xi\right) \\
&+\frac{1}{q}\left(\int_0^1\int_0^1 |\hat{o}(\kappa_1,\kappa_3)|^{\zeta\xi q}|\hat{o}(\kappa_1,\kappa_4)|^{\zeta(1-\xi)q}|\hat{o}(\kappa_2,\kappa_3)|^{(1-\zeta)\xi q}|\hat{o}(\kappa_2,\kappa_4)|^{(1-\zeta)(1-\xi)q}d\zeta d\xi\right) \\
&= \frac{1}{pe^{2p\alpha(\kappa_1+\kappa_3+\kappa_1+\kappa_4)}} \\
&+\frac{1}{q}\left(\int_0^1 L\left(|\hat{o}(\kappa_1,\kappa_3)|^{\xi q}|\hat{o}(\kappa_1,\kappa_4)|^{(1-\xi)q},|\hat{o}(\kappa_2,\kappa_3)|^{\xi q}|\hat{o}(\kappa_2,\kappa_4)|^{(1-\xi)q}\right)d\xi\right)
\end{aligned}$$

Because of the $L(\kappa_1,\kappa_2) < A(\kappa_1,\kappa_2)$ property, we can write

$$\begin{aligned}
&\left|\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\hat{o}(\tau_1,\tau_2)d\tau_1 d\tau_2\right| \\
&\leq \frac{1}{pe^{2p\alpha(\kappa_1+\kappa_3+\kappa_1+\kappa_4)}} \\
&\frac{1}{q}\left(\int_0^1 A\left(|\hat{o}(\kappa_1,\kappa_3)|^{\xi q}|\hat{o}(\kappa_1,\kappa_4)|^{(1-\xi)q},|\hat{o}(\kappa_2,\kappa_3)|^{\xi q}|\hat{o}(\kappa_2,\kappa_4)|^{(1-\xi)q}\right)d\xi\right) \\
&= \frac{1}{pe^{2p\alpha(\kappa_1+\kappa_3+\kappa_1+\kappa_4)}} \\
&+\left(\int_0^1 \frac{|\hat{o}(\kappa_1,\kappa_3)|^{\xi q}|\hat{o}(\kappa_1,\kappa_4)|^{(1-\xi)q}+|\hat{o}(\kappa_2,\kappa_3)|^{\xi q}|\hat{o}(\kappa_2,\kappa_4)|^{(1-\xi)q}}{2q}d\xi\right) \\
&= \frac{1}{pe^{2p\alpha(\kappa_1+\kappa_3+\kappa_1+\kappa_4)}} \\
&+\frac{L\left(|\hat{o}(\kappa_1,\kappa_3)|^{q}|\hat{o}(\kappa_1,\kappa_4)|^{q}\right)+L\left(|\hat{o}(\kappa_2,\kappa_3)|^{q}|\hat{o}(\kappa_2,\kappa_4)|^{q}\right)}{2q}.
\end{aligned}$$

Proof is the completed.

Corollary 3 *If we choose* $\alpha = 0$ *in Theorem 5, the result agrees* $AG-$*exponential convex on the coordinates with log-convexity on the coordinates.*

$$\left|\frac{1}{(\kappa_2-\kappa_1)(\kappa_4-\kappa_3)}\int_{\kappa_1}^{\kappa_2}\int_{\kappa_3}^{\kappa_4}\hat{o}(\tau_1,\tau_2)d\tau_1 d\tau_2\right|$$
$$\leq\frac{1}{p}+\frac{L\left(\left|\hat{o}(\kappa_1,\kappa_3)\right|^q\left|\hat{o}(\kappa_1,\kappa_4)\right|^q\right)+L\left(\left|\hat{o}(\kappa_2,\kappa_3)\right|^q\left|\hat{o}(\kappa_2,\kappa_4)\right|^q\right)}{2q}$$

where $\tau_1\in[\kappa_1,\kappa_2]$, $\tau_2\in[\kappa_3,\kappa_4]$ and $p^{-1}+q^{-1}=1$ dir.

REFERENCES

[1] Awan, M.U., Noor, M.A. and Noor, K.I., Hermite-Hadamard Inequalities for Exponentially Convex Functions, *Appl. Math. Inf. Sci.*, 12, No. 2, 405-409 (2018).

[2] Pejaric, J. E., Tong, Y. L. (1992). Convex functions, partial orderings, and statistical applications. Academic Press.

[3] Niculescu, C. P. (2000). Convexity according to the geometric mean. Math. Inequal. Appl, 3(2), 155-167.

[4] Akdemir, A. O. (2012). Inequalities of Ostrowski's type for m-and (alpha, m)-logarithmically convex functions via Riemann-Liouville Fractional integrals. arXiv preprint arXiv:1212.2165.

[5] Akdemir, A. O., Tunç, M. (2015). Ostrowski type inequalities for s-logarithmically convex functions in the second sense with applications. Georgian Mathematical Journal, 22(1), 1-7.

[6] Akdemir, A. O. , Ardıç, M. A., Özdemir, M. E. (2017). Inequalities via $s-$ convexity and log-convexity. Topological Algebra and Its Applications, 5(1), 74-85.

[7] Tomar, M., Erhan, S. E. T., Maden, S. (2015). Hermite-Hadamard type inequalities for log-convex stochastic processes. Journal of New Theory, (2), 23-32.

[8] Noor, M. A., Qi, F., Awan, M. U. (2013). Some Hermite-Hadamard type inequalities for log-h-convex functions. Analysis, 33(4), 367-375.

[9] Boche, H., Schubert, M. (2008). A calculus for log-convex interference functions. IEEE Transactions on Information Theory, 54(12), 5469-5490.

[10] Iscan, I., Kadakal, H., Kadakal, M. (2017). Some new integral inequalities for n-times differentiable log-convex functions. New Trends in Mathematical Sciences, 5(2), 10-15.

[11] Aslan, S., Akdemir, A. O. (2022, August). Exponentially convex functions on the co-ordinates and related integral inequalities. In Proceedings of the 8th International Conference on Control and Optimization with Industrial Applications (Vol. 2, pp. 120-122).
[12] S.S. Dragomir, On Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane, Taiwanese Journal of Math., 5, 2001, 775-788.

[13] S. Aslan, (2023). Eksponansiyel konveks fonksiyonlar için koordinatlarda integral eşitsizlikler. PhD thesis, Doktora Tezi, Ağrı İbrahim Çeçen Üniversitesi, Lisansüstü Eğitim Enstitüsü.

[14] Akdemir, A. O., Aslan, S., Dokuyucu, M. A., Çelik, E. (2023). Exponentially Convex Functions on the Coordinates and Novel Estimations via Riemann-Liouville Fractional Operator. Journal of Function Spaces, 2023.

[15] Aslan, S.,Akdemir, A. O. (2023). Exponential $s-$Convex Functions in the First Sense on the Co-ordinates and Some Novel Integral Inequalities. Turkish Journal of Science,8(2),85-92.

[16] Aslan, S., Akdemir, A. O., Dokuyucu, M. A. (2022). Exponentially $m-$ and $(\alpha,m)-$ Convex Functions on the Coordinates and Related Inequalities. Turkish Journal of Science, 7(3), 231-244.

[17] Akdemir, A. O., Aslan, S., Set, E. (2022, October). Some New Inequalities for Exponentially Quasi-Convex Functions ion the Coordinates and Related Hadamard Type Integral Inequalities. In 5th Internatonal Conference on Mathematical and Related Sciences Book of Proceedings (p. 120).

[18] Akdemir, A. O., Aslan, S., Ekinci, A. (2022, October). Some New Inequalities for Exponentially P-Functions on the Coordinates. In 5th InternatÄ±onal Conference on Mathematical and Related Sciences Book of Proceedings (94-108).

[19] Nie, D., Rashid, S., Akdemir, A. O., Baleanu, D., Liu, J. B. (2019). On some new weighted inequalities for differentiable exponentially convex and exponentially quasi-convex functions with applications. Mathematics, 7(8), 727.

[20] Rashid, S., Noor, M. A., Noor, K. I., Akdemir, A. O. (2019). Some new generalizations for exponentially s-convex functions and inequalities via fractional operators. Fractal and fractional, 3(2), 24.

[21] Rashid, S., Safdar, F., Akdemir, A. O., Noor, M. A., Noor, K. I. (2019). Some new fractional integral inequalities for exponentially m-convex functions via extended generalized Mittag-Leffler function. Journal of Inequalities and Applications, 2019, 1-17.

[22] Anderson, G.D., Vamanamurthy, M.K. and Vuorinen, M., Generalized convexity and inequalities, J. Math. Anal. Appl., 335 (2007), 1294-1308.

[23] Bullen, P.S., Handbook of Means and Their Inequalities, Math. Appl., vol. 560, Kluwer Academic Publishers, Dordrecht, 2003.

[24] Bullen, P.S., Mitrinovic, D.S. and Vasic P.M, Means and Their Inequalities, Reidel, Dordrecht, 1988.

Hash Functions and Key Exchange Protocols

Merve Beyza ÇUBUK[1] and Abdullah ÇAĞMAN[2]

[1]Ağrı İbrahim Çeçen University, Institute of Graduate Studies, Ağrı-TÜRKİYE
[2]Ağrı İbrahim Çeçen University, Faculty of Sicence and Letters, Department of Mathematics, Ağrı-TÜRKİYE

mbeyza2@gmail.com
acagman@agri.edu.tr

ABSTRACT

In this study, general information about hash functions is given, and the classification and usage areas of cryptographic hash functions are examined. Additionally, Diffie-Hellman style key exchange protocols and their working principles were analyzed.

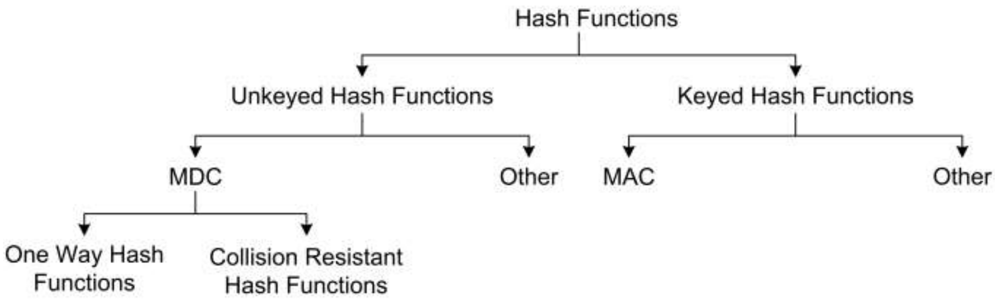
INTRODUCTION

Hash Functions

One of the most important issues of cryptographic systems is security. In messages sent between two parties, the recipient must understand the message and third parties must not be able to understand or decipher it. For this reason, functions that convert messages into an unreadable form are needed. Hash functions are one of them.
Hash functions produce a fixed-length output by processing data inputs with a mathematical calculation or algorithm. This process is called hash, and hashing is the process of converting data of any size into a fixed-size output.
In general, hash functions are used to ensure the integrity and security of data. Whether the data has been changed or not can be checked with hash functions. Additionally, the size of large data is reduced to a smaller size with a fixed length. Thus, the size of the data to be sent to the target is reduced.
Areas where the hash function is used are Digital Signature, Authentication and Blockchain.
There are two properties that a hash function has. These are;

1. Compression: Enter an input x of arbitrary finite bit length into n-bit. It converts the output to the constant h(x) of length.
2. Easy Computation: Given h and an input x, you can calculate h(x). It should be easy to calculate.

Cryptographic hash functions are generally divided into keyed and unkeyed hash functions. Keyed hash functions are divided into Message Authentication Codes (MACs) and other applications, and unkeyed hash functions are divided into Modification Detection Codes (MDCs) and other applications. In addition, MDCs are divided into two groups: One-Way Hash Functions and Collision-Resistant Hash Functions.



Key Exchange Protocols

Individuals are increasingly concerned about protecting their private conversations and online activity. This is why they want to encrypt their data using a cryptographic system. One of these cryptosystems is called a symmetric cryptosystem. The shared key is the cornerstone of symmetric cryptography, enabling both encryption and decryption of messages, but only if both parties possess it beforehand. Coordinating shared keys in symmetric cryptography poses logistical and security challenges, as they must be either securely exchanged beforehand or distributed via costly, protected channels. less to obtain and more expensive to maintain. Therefore, the communicating parties need a protocol to exchange keys. These protocols are often called key exchange protocols.

Key exchange protocols play a crucial role in securing modern communication. By enabling the secure establishment of shared keys, they underpin the confidentiality, integrity, and authenticity of sensitive information exchanged over digital networks. Their importance in safeguarding online privacy and security cannot be overstated.

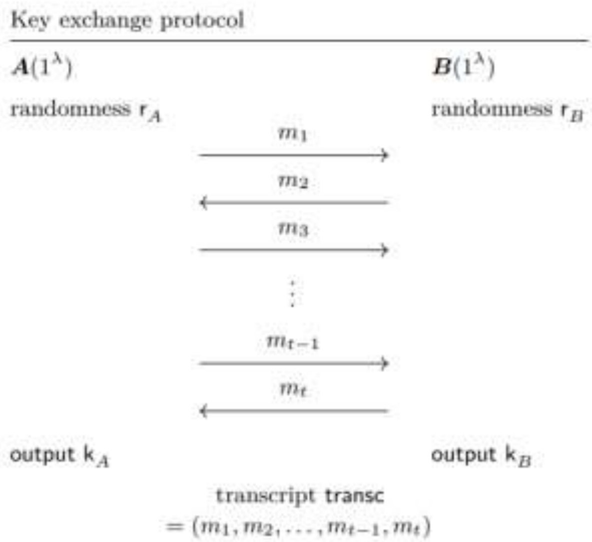
Key exchange protocols use mathematical approaches to obtain session keys for both sides without transferring them. This approach prevents anyone other than the legitimate parties from obtaining the session key by listening to the data transfer channel between the two sides.

Even when using a symmetric cryptosystem, the key exchange protocol must use a public-private key pair because the public-private key pair is used for session key calculations and not for encryption purposes. The key pair is created from distinctive domain parameters on both sides. These domain parameters and public keys are not secret. Only the private keys of both parties are secret. Whereas, because it uses a mathematical protocol, it is very difficult to renew the private key from the domain parameters and public key.

HASH FUNCTIONS AND KEY AGREEMENT

In the realm of cryptography, key agreement protocols are formally defined as interactive, two-party (A and B) processes that establish a shared secret key between them. The parties initiate the protocol with security parameter 1^{λ}, guiding their message exchanges towards key agreement. Additionally it has a random string r_A (for A) and r_B (for B) of sufficient length on both sides. Traditionally, we can say that A sends the first message m_1, B sends the second message m_2, A sends the third message m_3, and so on, B sends the last message m_t. We can safely assume that any necessary adjustments to align protocols with our format—such as A sending a redundant value in their first message or B doing so in their last—don't fundamentally alter their functionality. As long as the number of communication rounds remains within a polynomial range, these adjustments don't invalidate the impossibility result.

Each party's outgoing message is a function of their input, the transcript of previously received messages, and their local random tape. We formally model the protocol's execution as a sequence of algorithm invocations, with algorithms A(1^{λ},r_A, m_1, m_2, ...) and B(1^{λ},r_B, m_1, m_2, ...) taking the partial transcripts as input and generating the subsequent message as output, defining a deterministic state transition model. Protocol execution terminates with each party deriving and outputting local keys k_A and k_B , based on the shared transcript. We represent this process within our model by having algorithms A and B return both a key and a designated symbol. We denote by transc = (m_1, m_2,..., m_t) the transcript of the execution, consisting of the sequence of exchanged messages, where t is polynomial but may depend on the parties' inputs. The table below visually depicts the key exchange protocol's steps and interactions (Mittelbach, 2021).



To specify that the key exchange protocol involves parties A and B, we express it as $KE = \langle A, B \rangle$. Similarly, we define

$$(k_A, \text{transc}, k_B) \leftarrow \langle A(1^{\lambda}), B(1^{\lambda}) \rangle$$

a random variable (symbol) to model the outcome of a key exchange protocol execution, with its probability distribution dependent on the random tapes of r_A and r_B. The outcome captures the keys and the transcript, revealing both the final results and the path taken to reach them. If we consider the execution for fixed random tapes r_A and r_B. We get a deterministic output,

$$(k_A, \text{transc}, k_B) \leftarrow \langle \mathrm{A}(1^{\lambda}; r_A), \mathrm{B}(1^{\lambda}; r_B) \rangle$$

While we haven't yet specified requirements for the parties' keys k_A and k_B, successful key exchange protocols ensure they always match (k_A = k_B), enabling reliable communication using a shared key. This property is known as perfect correctness. Accepting a negligible error rate allows for more efficient protocols, but necessitates careful analysis of potential security implications. We'll restrict our attention to perfectly correct protocols to simplify the exposition of our impossibility findings. Having established perfect correctness, we must now move on to the cornerstone of all cryptographic protocols: security. The paramount objective of a key exchange protocol is to safeguard the shared key, ensuring it's accessible only to A and B. The protocol's security model formally captures the threat of an eavesdropping adversary E (Eve) who passively observes the communication channels during the key exchange. Even with the full transcript in hand, Eve (the adversary) shouldn't be able to glean any information about A's key k_A, like its value or structure. Under the assumption of almost-sure key identicality, we can model the adversary's goal as predicting key k_A, as any successful prediction of key k_B would also reveal key k_A with overwhelming probability.

Diffie-Hellman Key Exchange Protocol

Diffie-Hellman's security hinges on the intractability of discrete logarithms in finite fields. The discrete logarithm can be defined as:
"Given a prime p, a generator α of Z_P^* and an element $\beta \in Z_P^*$, find the integer x, $0 \leq x \leq p-2$, such that $\alpha x \equiv \beta$" (Akalp, 2008).
Now let us explain the Diffie-Hellman key agreement protocol by making use of this definition. Parties A and B begin by establishing a common mathematical foundation: a group G, its generators g, and security parameter λ, ensuring a shared understanding for secure communication. By selecting a prime number q as the group's order, we introduce a level of mathematical elegance and security that can enhance the robustness of the cryptographic protocol. Now each side chooses a secret power x for A and y for B. Then it sends g^x→X and g^y→Y to the other party respectively. Then, in the group, A computes the y^x→k_A and B computes it as x^y →k_B. Let's not forget that for well-formed data we have equality A which means both parties agree on the same value.

$$k_A = Y^x = (g^y)^x = g^{xy} = (g^x)^y = X^y = k_B \quad \ldots\ldots\ldots \text{(A)}$$

It's essential to note that the key derivation function doesn't simply equate k_A = k_B. Instead, it leverages the shared Diffie-Hellman value as a starting point to meticulously construct a key with desirable cryptographic properties, including a uniform bit distribution. Diffie-Hellman protokolü, güvenliğini ustaca görünüşte basit ama son derece zorlayıcı bir matematiksel bilmeceye dayandırıyor: ayrık logaritma problemi. Doğasında var olan zorluk, yetkisiz anahtar alımına karşı zorlu bir bariyer oluşturarak iletişimin gizliliğini sağlar. That is, if an adversary can compute the discrete logarithm of X or Y, he can also compute the shared key of both parties. However, it is not known whether the discrete logarithm assumption is also sufficient to demonstrate security. The Diffie-Hellman protocol's security rests upon a crucial assumption: the intractability of the discrete logarithm problem. While no attacks have exploited this assumption directly, the possibility of unforeseen weaknesses within the protocol itself cannot be entirely dismissed. Although assuming Diffie-Hellman's security is reasonable based on its enduring resistance to attacks, caution dictates ongoing vigilance and a readiness to adapt should novel vulnerabilities emerge. So it should not be possible to calculate g^{xy} from g^x and g^y.

The Diffie-Hellman protocol ingeniously leverages intricate mathematical properties of the underlying number-theoretic structure to establish secure key exchange. Due to the mathematical harmony within the cyclic group G, the order of exponentiation doesn't affect the outcome, enabling A and B to independently reach a shared secret key. The current approach offers limited guidance for designing secure key exchange protocols using random oracles when those crucial mathematical properties are absent, highlighting a significant challenge for cryptographic innovation.

Hash-based Key Agreement

Recently, studies on the use of hash functions in key agreement protocols have gained intensity (Wang, 2021; Lee 2010; Guo, 2010; Yoon, 2011).

REFERENCES

Mittelbach, A., & Fischlin, M. (2021). The theory of hash functions and random oracles. An Approach to Modern Cryptography, Cham: Springer Nature.

E. Akalp, The Web-Based Application of Key Exchange Protocols For Digital and Mobile Signatures, In Computer Software, September 2008, İzmir, page 1-6.

Wang, J., Zhu, Y., & Maqbool, S. (2021). An efficient hash-based authenticated key agreement scheme for multi-server architecture resilient to key compromise impersonation. Digital Communications and Networks, 7(1), 140-150.

Lee, J., Kim, I. T., & Park, J. B. (2010). A Study on a Group Key Agreement using a Hash Function. Journal of the Korea Institute of Military Science and Technology, 13(4), 627-634.

Guo, X., & Zhang, J. (2010). Secure group key agreement protocol based on chaotic hash. Information Sciences, 180(20), 4069-4074.

Yoon, E. J., & Yoo, K. Y. (2011). Cryptanalysis of group key agreement protocol based on chaotic hash function. IEICE TRANSACTIONS on Information and Systems, 94(11), 2167-2170.

Some Cryptographic Applications of Hash Functions

Ömer Faruk ÇETİN[1] and Abdullah ÇAĞMAN[2]

[1]*Ağrı İbrahim Çeçen University, Institute of Graduate Studies, Ağrı-TÜRKİYE*
[2]*Ağrı İbrahim Çeçen University, Faculty of Science and Letters,*
Department of Mathematics, Ağrı-TÜRKİYE

omerfarukcetin18@hotmail.com
acagman@agri.edu.tr

ABSTRACT

Our aim in this study is to investigate hash functions and their various applications in cryptography. For this purpose, some information is given about the properties of traditional hash functions. Additionally, the types of attacks that these functions may encounter are also mentioned. Potential vulnerabilities and risks associated with hybrid functions were examined and information was provided regarding the vulnerabilities they encountered.

INTRODUCTION

Cryptography is concerned with the formulation and creation of robust cryptosystems and ciphers to ensure secure communication. The discipline focuses on developing intricate mechanisms that safeguard sensitive information from unauthorized access or interception.
Cryptanalysis is the practice of systematically examining and analyzing cryptosystems and ciphers, with the primary objective of deciphering or "cracking" them. This specialized field focuses on a thorough investigation of cryptographic algorithms, aiming to identify and exploit potential weaknesses to gain unauthorized access to encrypted data.
Cryptography and cryptanalysis come together to form cryptology. Cryptology is dedicated to enabling secure communication within insecure channels. It involves the development of methods and systems to safeguard information integrity and confidentiality when transmitted over channels susceptible to unauthorized access or interception.
As early as 1900 BC, cryptography has a historical footprint, with ancient Egypt employing rudimentary encryption methods to maintain message confidentiality. In this era, cryptographic practices included altering the letters or words within a message and concealing specific parts of the communication to ensure secrecy.
Around 100 BC, Julius Caesar devised a cipher, now named after him, where each letter in the message is replaced by the next three letters in the alphabet.[1]
In the 5th century BC, Spartans employed permutation ciphers for secure message transmission.
In military and diplomatic communications of the Middle Ages, cryptography gained extensive usage.
With the rise of computers in the 19th century, cryptography entered a new era, experiencing a significant evolution.

[1]History of cryptography - Wikipedia

In 2008, Bitcoin emerged onto the scene as a cryptocurrency, its foundation resting on the SHA-256 hash function.
Cryptography is a field in constant flux, with the continuous emergence of innovative algorithms and techniques. Future developments in cryptography will be strongly influenced by the progress of technologies such as artificial intelligence, quantum computers, and blockchain.

CRYPTOGRAPHIC HASH FUNCTIONS

Definition 1. A cryptographic hash function is a function that takes bit strings of arbitrary length and converts them into bit strings of a specific (n bits) length.
Cryptographic hash functions are expected to provide the following properties.

1. Front Image Durability: Given an output y, it should be difficult to find an input x that gives that output.
2. Second Front Image Durability: Given an output y and an input x_1 that satisfies $h(x_1) = y$, it should be difficult to find a second input x_2 that satisfies $h(x_2) = y$.
3. Collision Resillience: It should be difficult to find inputs x_1 and x_2 that satisfy the equation $h(x_1) = h(x_2) = y$.

In addition, the outputs should be random.
Hash functions are used extensively in cryptography. Some of them are :

Digital Signatures
Message Verification Codes
Random Number Generators
Key Generation Functions

ITERATIVE HASH FUNCTIONS

Definition 2. Several keyless hash functions follow an iterative procedure, where inputs of varying lengths are condensed by subjecting fixed-size blocks of the input to consecutive operations. An iterative hash function is formulated through a compression function, denoted as f, which transforms a (t + n)-bit input into an n-bit output.[2]

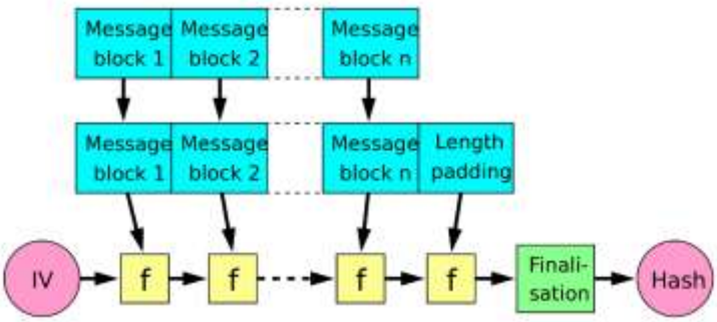


Figure 1 Merkle-Damgard Structure [3]

[2] S. Lucks, "Design Principled for Iterated Hash Functions", in IACR Cryptology ePrint Archive, 2004, pp.253.
[3] Illustration by David Göthberg, Sweden.

Until Sha-3, all hash functions are constructed using the Merkle-Damgard structure. The process begins by initializing the function with an initialization vector, taking the first message block as input. Subsequently, the function iterates, taking the output along with the next message block as inputs, producing a new output. This loop continues until all blocks are processed, and the resulting output is termed the summary code.

KEYLESS HASH FUNCTIONS

Definition 3. MDC keyless cryptographic hashing designed to ensure data integrity function resistance to which of the three problems defined in the previous section shows that one way hash functions (one way hash functions) (OWHF) and as collision resistant hash functions (CRHF) is divided into two.

1. The definition of the hash function H is well known.
2. Given x, $H(x)$ is easy to calculate.
3. The front image must have durability.
4. The second front image must have durability.
5. It must be collision resistant.

Functions that satisfy the first 4 properties are called one-way hash functions (OWHF), while functions that satisfy all properties are called collision-resistant hash functions (CRHF).

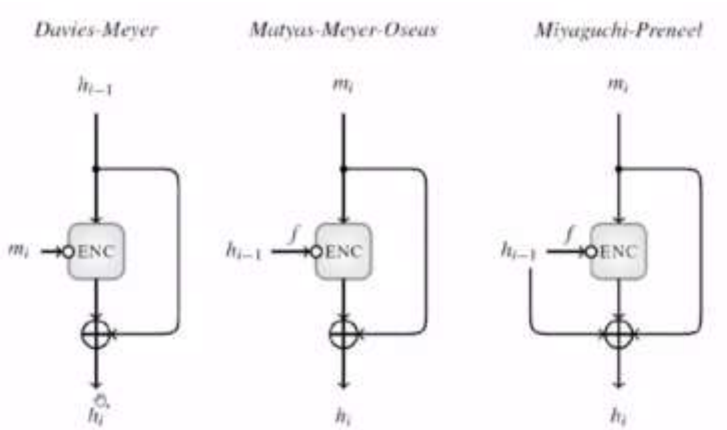


Figure 2 Davies-Meyer, Matyas-Meyer-Oseas, Miyaguchi-Preneel (Illustration by David Göthberg, Sweden.)

Hash functions can also be generated using block ciphers, and in the literature, three of them are acknowledged for their security. Take the Davies-Meyer method as an example: consider the initial vector as a plaintext block, treat the first message block to be summarized as the key, encrypt it, and output the resulting ciphertext block. If there's a second block, use this output as a new plaintext block. Repeat this process until all message blocks are processed to obtain the final version.

Conventional Hash Functions

MD4

It was designed by Ronald Rivest in 1990.
Block Length: 512 bits.
Hash Length: 128 bits.
Number of Cycles: 48
Type: Merkle-Damgard
A,B,C,D are 32-bit state values and m_i is part of your message.
Collision Resilience: Collision generation takes microseconds.
Front Image Durability: 2^{95} operations. [4]
It has influenced many subsequent designs. MD5, SHA-1, RIPEMD

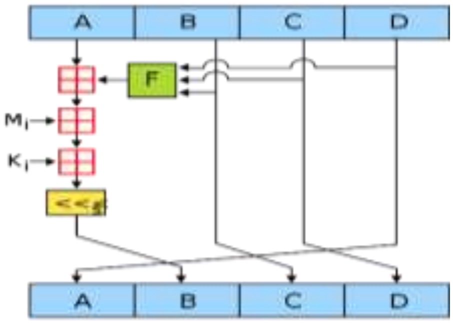


Figure 3 MD4 (Illustration by Matt Crypto)

MD5

It was designed by Ronald Rivest in 1991 as a replacement for MD4.
Block Length: 512 bits
Hash Length: 128 bits
Number of Cycles: 64
Type: Merkle-Damgard
A,B,C,D are 32-bit state values and m_i is part of your message.

Each of the 16 loops uses F, G, H, I as the f function.

- F(B, C, D) = (B ∧ C) ∨ (¬B ∧ D)
- G(B, C, D) = (B ∧ D) ∨ (C ∧ ¬D)
- H(B, C, D) = B ⊕ C ⊕ D
- I(B, C, D) = C ⊕ (B ∨ ¬ D)

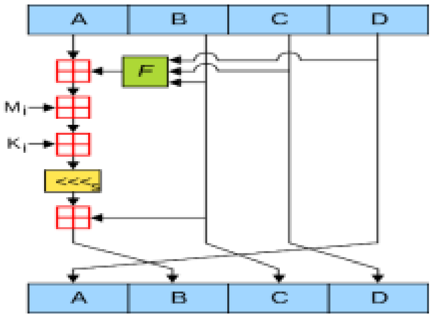


Figure 4 MD5 (Illustration by Surachit)

Collision Resilience: It takes 2^{18} operations to generate a collision (less than 1 second)
Front Image Durability: $2^{123.4}$ operations [5]

2004: It is shown how to create two different files that will give the same MD5 digest.
2005: Lenstra-Wang-Weger demonstrated how to generate X.509 certificates with two different public keys with the same MD5 hash.
2008: It was shown that it was possible to create fake SSL certificates and certificate authorities were advised to stop using MD5.
2012: Flame malware used forged Windows code signing certificates using MD5 conflicts.

[4] R. Rivest, "The MD4 Message Digest Algorithm", IETF RFC 1320, 1992.
[5] R. Rivest, "The MD5 Message Digest Algorithm", IETF RFC 1321, 1992.

```
d131dd02c5e6eec4 693d9a0698aff95c 2fcab58712467eab 4004583eb8fb7f89
55ad340609f4b302 83e488832571415a 085125e8f7cdc99f d91dbdf280373c5b
d8823e3156348f5b ae6dacd436c919c6 dd53e2b487da03fd 02396306d248cda0
e99f33420f577ee8 ce54b67080a80d1e c69821bcb6a88393 96f9652b6ff72a70
```

```
d131dd02c5e6eec4 693d9a0698aff95c 2fcab50712467eab 4004583eb8fb7f89
55ad340609f4b302 83e4888325f1415a 085125e8f7cdc99f d91dbd7280373c5b
d8823e3156348f5b ae6dacd436c919c6 dd53e23487da03fd 02396306d248cda0
e99f33420f577ee8 ce54b67080280d1e c69821bcb6a88393 96f965ab6ff72a70
```

Figure 5 MD5 Collision

Most places are the same, differences are shown in blue. Both inputs generate the same hash value.

SHA-1
It was designed by the NSA in 1993.
This version is now known as SHA-0.
In 1995 it was published as SHA-1 with only one bit operation rotation.
Block length: 512 bits
Hash length: 160 bits
Number of cycles: 80
Type: Merkle-Damgard [6]

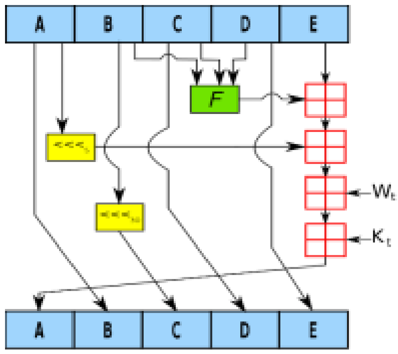


Figure 6 SHA-1

Years ago, academic texts mentioned that a collision could be found, but it was not possible to do so with the processing power available at the time. In 2017, Marc Stevens and his team at Google, using their graphics cards, found a conflict for SHA-1.

Figure 7 SHA-1 Collision

The message consists of 2 blocks. The differences are shown in red and blue. When you process both of them with SHA-1, it gives the same output.

[6] FIPS180-3, Secure Hash Standard (SHS), National Institute of Standards and Technology, US Department of Commerce, Washington D. C., 2008.

SHA-2
Prepared by the NSA in 2001.
Block length: 512 veya 1024 bits
Hash length: 224, 256, 384 veya 512 bits
Number of cycles: 64 veya 80
Type: Merkle-Damgard

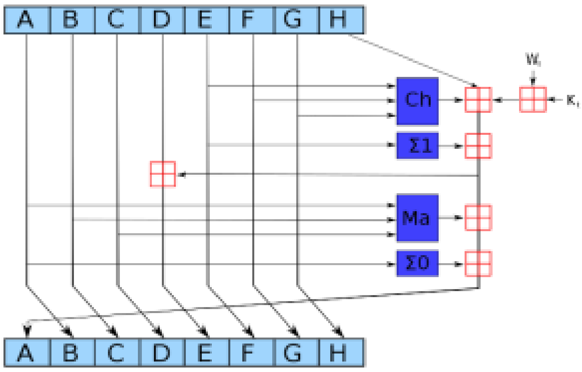


Figure 8 SHA-2 (Illustration by Kockmeyer)

SHA-3 Competition
All functions before Sha-2 are broken. The only function we have left is the Sha-2 function. It's very similar to the previous designs. So NIST thought we should have a plan b, we should have a second summary function. And organized a competition. 4 of the 64 applicants were Turkish designers. Keccak algorithm won the competition. So the Keccak algorithm was named Sha-3.

- Designers: Guido Bertoni, Joan Daemen, Michael Peeters and Gilles Van Assche
- Block length: 1152, 1088, 832 or 576 bits
- Hash length: 224, 256, 384 or 512 bits
- Number of cycles: 64 or 80
- Type: Sponge

KEYED HASH FUNCTIONS (MACs)

Definition 3. Keyed hash whose purpose is to guarantee the integrity of the message, to prove the message functions are also called Message Authentication Code algorithms.
The sender and receiver of a message need to agree on the same key before initiating communication.

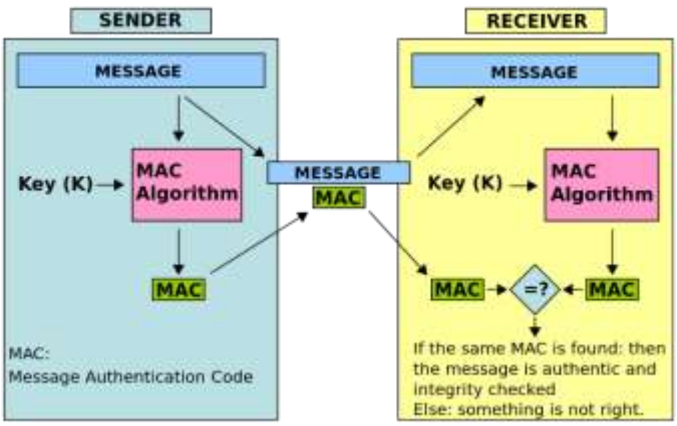


Figure 9 MACs (Illustration by Twisp)

To protect the message, the sender calculates the MAC of the m - bit length bit stripe appropriate to the message and appends this stripe to the message. MAC is a complex function of each bit of the message and the key. On receipt of the message, the receiver recalculates the MAC and verifies whether it corresponds to the MAC value sent. [7]

CBC-MAC

The most widely used MAC algorithm is CBC-MAC, which is based on a block cipher that enables the use of cipher block chaining (CBC-Cipher Block Chaining).
Block length: 64 bits (E-block password)
Key: 56-bit DES key

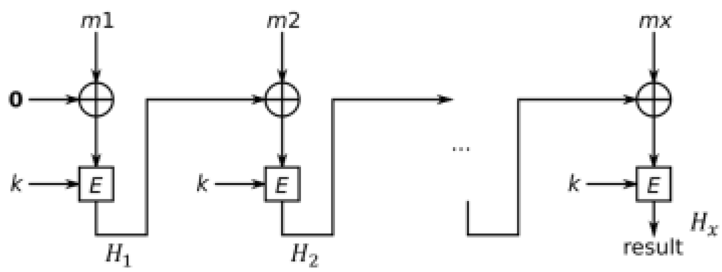


Figure 10 CBC-MAC[8]

If necessary, m is filled. The padded text is divided into n-bit blocks m_1, m_2,..., m_x. The H_1 blog is computed using the key k and E. MAC is the n-bit H_x block.

TYPES OF ATTACK

Dictionary Attack: Precomputed summaries of words from a dictionary and numerical combinations are calculated and systematically stored in a table. If an attacker identifies matching summary values in the database and their own table, they can gain access to the corresponding passwords.
Solution: *Instead of the password digest, a randomly generated salt value is appended to the end of the password and the password // salt digest is stored in the database. The attacker would therefore need to create a separate table for each salt value, but this is not possible due to computational and memory requirements.*

Brute Force Attack: Once the digest value is captured, each password is tried one by one. This works well when users choose short and easy-to-remember passwords.
Solution: *Instead of using the digest of a password, we can use the last digest value obtained by repeatedly passing that digest output to the digest function. For example, if we do this 1000 times, the attacker's job will be 1000 times harder.*

REFERENCES

History of cryptography - Wikipedia
S. Lucks, "Design Principled for Iterated Hash Functions", in IACR Cryptology ePrint Archive, 2004, pp.253.

[7] Mironov, "Hash Functions: Theory, Attacks, and Applications", Microsoft Research, Silicon Valley Campus, 2005, pp.10-11.
[8] B. V. Rompay, "Analysis and Design of Cryptographic Hash functions, MAC algorithms and Block Ciphers", Ph.D. thesis, Electrical Engineering Department, Katholieke Universiteit, Leuven, Belgium, 2004.

R. Rivest, "The MD4 Message Digest Algorithm", IETF RFC 1320, 1992.

R. Rivest, "The MD5 Message Digest Algorithm", IETF RFC 1321, 1992.

FIPS180-3, Secure Hash Standard (SHS), National Institute of Standards and Technology, US Department of Commerce, Washington D. C., 2008.

Mironov, "Hash Functions: Theory, Attacks, and Applications", Microsoft Research, Silicon Valley Campus, 2005.

B. V. Rompay, "Analysis and Design of Cryptographic Hash functions, MAC algorithms and Block Ciphers", Ph.D. thesis, Electrical Engineering Department, Katholieke Universiteit, Leuven, Belgium, 2004.

Some Analysis for the Maximal C_6 Class of Classical Groups with Dimensions 6 and 8

Zehra ÇELİK ÇÖP[1], Abdullah ÇAĞMAN[2] and Kadirhan POLAT[2]

[1]*Institute of Graduate Studies,*
Ağrı İbrahim Çeçen University, Ağrı-TURKEY
[2]*Department of Mathematics, Faculty of Science and Arts,*
Ağrı İbrahim Çeçen University, Ağrı-TURKEY

zcelikcop@gmail.com
acagman@agri.edu.tr
kadirhanpolat@agri.edu.tr

ABSTRACT

In this notes Aschbacher's Thorem will be given and the Aschbacher's classical group classes will be defined. We will do some reduction algorithm analysis for the maximal C˙6 class of classical groups with dimensions 6 and 8.

INTRODUCTION

In this section, we present some preliminaries.
Let $n \in N$ and $\mathbb{F}_q$ be the field with $q = p^e$ elements. Let $V \coloneqq \mathbb{F}_q^{1\times n}$ be the $\mathbb{F}_q$-vector space of row vectors.

Theorem 1 (Aschbacher 1984): Let $G \leq GL_n(\mathbb{F}_q)$ and $n \geq 2$. Then G lies in at least one of the classes $\mathcal{C}_1$ to $\mathcal{C}_9$ of subgroups of $GL_n(\mathbb{F}_q)$ [1].

One of the classes of Aschbacher's Theorem is C_6 and definition of this clas as follows:

Class $\boldsymbol{\mathcal{C}_6}$: $G \leq GL_n(\mathbb{F}_q)$ lies in $\mathcal{C}_6$ if the natural module V is irreducible $n = r^m$ for a prime r and either r is odd and G has a normal subgroup E that is an extraspecial r -group of order $r^{(1+2m)}$ and exponent r ,or $r = 2$ and G has a normal subgroup E that is either extraspecial of order 2^{1+2m} or a central product of a cyclic group of order 4 with an extraspecial group of order $2^{1+2m}$and in both cases the linear action of G on the $\mathbb{F}_r$-vector space $E/Z(E)$ of dimension $2m$ is irreducible.

This class is in practice computationally under control.

A reduction algorithm and analyses for $n = r^2$ is given for $\mathcal{C}_6$ class in [3] and some other analyses for $n = r^3$ is given in [4]. In this note, we give some analyses of this algorithm for r^3 and r^4. Fort his purpose, some preliminaries as follows:

Definition 1: Let G and H be groups and let's assume that G has an action on H. Depending on this action, the commutator subgroup is defined as:

$$\Gamma_G(H) = \langle g^h\, kh^{-1}\, k^{-1} \mid g \in G \text{ ve } h, k \in H \rangle$$

where g^h is the action of g on h [8].

Definition 2: Let G be a group. The lower central series $\{\Gamma_i(G)\}_{(i\in\mathbb{N})}$ of G is defined inductively as follows: $\Gamma_1(G) = G$ and for all $i \in \mathbb{N}$, $\Gamma_{i+1}(G)=[G, \Gamma_i(G)]$. Similarly, the derived series $\{G^{(i)}\}_{i\in\mathbb{N}\cup\{0\}}$ of G is defined as $G^{(0)} = G$ and $G^{(i)} = [G^{(i-1)}, G^{(i-1)}]$ for every $i \in \mathbb{N}$. Note here that $\Gamma_2(G)= G^{(1)} = G'$ [9].

For any group-theoretic property $\mathcal{P}$ if there exist a group H with property $\mathcal{P}$ for any nontrivial element $x \in G$ and a surjective homomorphism $\varphi: G \to H$ such that $\varphi(x) \neq 1$, G is said to be residually $\mathcal{P}$. It is well known that $\bigcap_{i\geq 1}\Gamma_i(G) = \{1\}$ (respectively $\bigcap_{i\geq 0} G^{(i)} = \{1\}$ is a necessary and sufficient condition for a group G to be residually nilpotent (resp. residually solvable).

Proposition 1: Let G and H be groups, and let $\varphi: G \to Aut(H)$ be an action of G on H. Let $\widehat{H}$ be the subgroup of H generated by elements of the form $\varphi(g)(h) \cdot h^{-1}$, where $g \in G$ and $h \in H$, and let L be the subgroup of H generated by $\Gamma_2(H)$ and $\widehat{H}$. In this case, φ induces an action of $\Gamma_2(G)$ on L, also denoted by φ, and $L \rtimes_\varphi \Gamma_2(G) = \Gamma_2(H \rtimes_\varphi G)$. Specifically, $\Gamma_2(H \rtimes_\varphi G)$ is the subgroup generated by $\Gamma_2(H)$, $\Gamma_2(G)$ and $\widehat{H}$ [9].

Reminder 1: The group L is actually the commutator subgroup $\Gamma_G(H)$.

Proposition 2: If $p > 3$, the derived subgroup of $SL(2, p)$ is equal to itself [6].

Proposition 3: If $n = 2, SL(n, p) = Sp(n, p)$ [6].

Proposition 4: If $p = 3$ and $n \geq 2$, the derived subgroup of $Sp(2n, p)$ is equal to itself [6].

Proposition 5: If $p \geq 4$, the derived subgroup of $Sp(2n, p)$ is equal to itself [6].

Proposition 6: If $n \geq 3$ or $n = 2$ and $p \geq 3$, the derived subgroup of $GL(n, p)$ is equal to itself [7].

Theorem 2: Let G be a group, and let $D_1, D_2, \ldots, D_n$ be its subgroups. If $G = D_1 \times D_2 \times \ldots \times D_n$ the solvable residue of G is the direct product of the solvable residue of subgroups D_i $(1 \leq i \leq n)$ [5].

MAIN RESULTS

The following auxiliary theorem in [3] forms the basis for the analyses.

Lemma 1: Let H be a finite group and A be the elementary abelian normal r (r prime) subgroup of H. In this case the following is provided:

Let $r > 2$ and let $c \in H$ be assumed to be a fixed element that induces-Id on A. Let b be any element of the side set cA. In this case, for a uniformly distributed random element $h \in H$ and any integer k

$$Prob\ (h^k = b|\ \ h^k \in cA) = 1/|A|.$$

Let $b \in H$ be a fixed element with a nontrivial effect on A. In this case for a fixed side set C of A in H and a uniformly distributed random $h \in C$

$$Prob\ ([b,h] \neq 1|\ \ [b,h] \in A) \geq 1 - 1/r.$$

One of the maximal subgroups of Sp(6,3) is $GU(3,3).2$ and solvable residue of this subgroup is $SU(3,3)$ [2].

The theorem involving the analysis for this perfect subgroup is as follows:

Theorem 3: If $\bar{G} = G/(R \cap G) \cong SU(3,3)$, BlindDescent [3] produces a suitable x element with probability greater than $1 - \delta$ after processing $750log(1/\delta), y$ elements.

Proof: We have $r = 3$ and so,

$$Prob([x,y] \in R \cap G) \geq 1/500.$$

Also, it is obtained from Lemma 1 that

$$Prob([x,y] \notin Z(R \cap G)|[x,y] \in R \cap G) \geq 2/3.$$

Thus,

$$Prob([x,y] \notin Z(R \cap G) \wedge [x,y] \in R \cap G) \geq 1/750.$$

Theorem 4: $\bar{G} = G/(R \cap G) \cong PSL(2,17)$, BlindDescent finds a suitable x with probability greater than $1 - \delta$ after processing $750log(1/\delta)$, y elements.

Proof: We have $r = 2$ and so,

$$Prob([x,y] \in R \cap G) \geq 1/280.$$

Also, it is obtained from Lemma 1 that

$$Prob([x,y] \notin Z(R \cap G)|[x,y] \in R \cap G) \geq 1/2.$$

Hence,

$$Prob([x,y] \notin Z(R \cap G) \wedge [x,y] \in R \cap G) \geq 1/560.$$

ACKNOWLEDGEMENT

This work was supported by TÜBİTAK (Project number: 121F477).

REFERENCES

1. Aschbacher, M., 1984. On the maximal subgroups of finite classical groups. Invent.Math., 76(3), 469-514.
2. Bray, J., Holt, D. and Rooney-Dougal, C., 2013. The maximal subgroups of the low dimensional finite classical groups. Cambridge University Press, 438.

3. Brooksbank, P. and Niemeyer, A.C. and Seress, A., 2006. A reduction algorithm for matrix groups with an extraspecial normal subgroup. Finite Geometries, Groups and Computation, deGruyter, 1-16.
4. Çağman, A. and Ankaralıoğlu, N. (2016). A contribution to the analysis of a reduction algorithm for groups with an extraspecial normal subgroup. Turkish Journal of Mathematics, 40(4), 914-923.
5. Doerk, K. and Hawkes, T., 1978. On the residual of a direct product. Arch. Math. (Basel), 30, 458-468.
6. Grove, L.C., 2002. Classical Groups and Geometric Algebra. American Mathematical Society, 169p, Providence, Rhode Island, USA.
7. Rotman, J.J., 1995. An Introduction to the Theory of Groups. Springer Verlag, 513p, New York, USA.
8. P. Hilton, G. Mislin and L. Roitberg, 1975. Localization of nilpotent groups and spaces, NorthHolland Mathematics Studies, No. 15, Notas de Matem´atica, No. 55, North-Holland Publishing Co., Amsterdam, 1975. MR0478146 (57:17635).
9. Gonçalves, D., & Guaschi, J. (2009). The lower central and derived series of the braid groups of the sphere. *Transactions of the American Mathematical Society*, *361*(7), 3375-3399.

Gröbner Bases and Applications of Toric Ideals

Merve EKER[1] and Abdullah ÇAĞMAN[2]

[1]Ağrı İbrahim Çeçen University, Institute of Graduate Studies, Ağrı-TÜRKİYE
[2]Ağrı İbrahim Çeçen University, Faculty of Science and Letters,
Department of Mathematics, Ağrı-TÜRKİYE

merve9eker@gmail.com
acagman@agri.edu.tr

ABSTRACT

In this paper, we study on Gröbner bases of toric ideals and some of their applications. After discussing the basic issues regarding Göbner bases and toric ideals, some applications of Gröbner bases of toric ideals as integer programming, triangulations of convex polytopes and contingency tables (statistics) are mentioned.

INTRODUCTION

A Gröbner basis is a set of multivariate nonlinear polynomials that allows simple algorithmic solutions for many fundamental problems related to algebraic and applied fields. Formal definition of Gröbner basis and some related concepts are as follows [1]:

Definition 1. Let I be an ideal and $G = \{g_1, g_2, \dots, g_t\}$ be a set of nonzero polynomials of I. Then, G is a Gröbner basis if and only if for all $f \in I$ such that $f \neq 0$, there exists $i \in \{1,2, \dots, t\}$ such that $lp(g_i)$ divides $lp(f)$.

Theorem 1 (Division Algorithm): Given $a, b \in k[x]$ (a and b are elements of one variable polynomial rings) with $b \neq 0$, there exists unique $q, r \in k[x]$ with $r = 0$ or $deg(r) < deg(b)$ such that;

$$a = bq + r.$$

We can use the division algorithm to find the greatest common divisor of two polynomials with Euclidean Algorithm.

Theorem 2 (Euclidean Algorithm): For $a, b \in k[x]$, $b \neq 0$, $(a, b) = (rn)$ where rn is the last non-zero remainder in the sequence of divisions

$$a = bq_1 + r_1,$$

$$b = r_1 q_2 + r_2,$$

$$r_1 = r_2 q_3 + r_3,$$

.
.
.

$$r_{n-2} = r_{n-1}q_n + r_n$$

$$r_{n-1} = r_n\, q_n + 0$$

Here, $r_n = xa + yb$ for $x, y \in k[x]$ can be calculated explicitly (by solving the above equations). We can use these algorithms to decide things like ideal membership (when $a \in (a_1, .., a_k)$) and equality (when $(a_1, .., a_k) = (b_1, .., b_k)$). Above, as the algorithms progress, they produce smaller degree polynomials at each step, ending with a remainder of zero. To extend these ideas to polynomials in various variables, the concept of the size of polynomials is needed. This is possible with term orders.

Definition 2 (Monomial Order). Let T_n be a set of all monomials in the variables $x_1, \dots, x_n$. A total order $<$ on T_n is called a monomial order if $<$ satisfies the following:

- u is element of T_n; $u \neq 1$ and $1 < u$.
- u, v, w is element of T_n; $u < v$ and $uw < vw$.

There are different sorting methods to find the monomial order.

Lexicographic order: It can be define the lexicographical order on T_n with $x_1 > \cdots > x_n$ as follows: For

$$\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n), \boldsymbol{\beta} = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$$

we define

$$\boldsymbol{x}^{\alpha} < \boldsymbol{x}^{\beta} \Leftrightarrow \begin{cases} \text{the first coordinates } \alpha_i \text{ and } \beta_i \text{ in } \boldsymbol{\alpha} \text{ and } \boldsymbol{\beta} \\ \text{from the left, which are different, satisfy } \alpha_i < \beta_i. \end{cases}$$

Degree Lexicographic order: **:** It can be define the lexicographical order on T_n with $x_1 > \cdots > x_n$ as follows: For

$$\alpha = (\alpha_1, \dots, \alpha_n), \beta = (\beta_1, \dots, \beta_n) \in N^n$$

we define

$$\boldsymbol{x}^{\boldsymbol{\alpha}} < \boldsymbol{x}^{\boldsymbol{\beta}} \Leftrightarrow \begin{cases} \sum_{i=1}^{n} \alpha_i < \sum_{i=1}^{n} \beta_i \\ or \\ \sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \beta_i \text{ and } \boldsymbol{x}^{\boldsymbol{\alpha}} < \boldsymbol{x}^{\boldsymbol{\beta}} \\ \text{with respect to lex with } x_1 > x_2 > \cdots > x_n. \end{cases}$$

TORIC IDEALS AND APPLICATIONS

Fix a subset $\mathcal{A} = \{a_1, a_2, \dots, a_n\} \subset \mathbb{Z}^d \backslash \{0\}$. Let π a semigroup homomorphism as follows:

$$\pi: \mathbb{N}^n \to \mathbb{Z}^d,\ \ u = (u_1, \dots, u_n) \mapsto \sum_{i=1}^{n} a_i u_i = Au.$$

Then, $\pi(\mathbb{N}^n) = \{Au: u \in \mathbb{N}^n\}$ is called the monoid generated by $\mathcal{A}$. The semigroup ring of $\mathbb{N}^n$ is $k[x] = k[x_1, \dots, x_n]$, and $\mathbb{Z}^d$ is the Laurent ring $k[t^{\pm 1}] \coloneqq k[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$. So, it is induced by π that

$$\hat{\pi}: k[x] \to k[t^{\pm 1}], x_j \mapsto t^{a_j} \coloneqq t_1^{a_{1j}} t_2^{a_{2j}} \dots t_d^{a_{dj}}.$$

Definition 3. The toric ideal of $\mathcal{A}$ is the kernel of the map $\hat{\pi}$ which is denoted as $I_{\mathcal{A}}$ [6].

Some applications of Gröbner basis of Toric ideals are as follows:

- Integer programming ([2]).
- Triangulations of convex polytopes ([3],[4]).
- Contingency tables (statistics) ([5]).

REFERENCES

1. Adams, W. W., & Loustaunau, P. (2022). An introduction to Gröbner bases (Vol. 3). American Mathematical Society.
2. Conti, P., & Traverso, C. (1991, October). Buchberger algorithm and integer programming. In International Symposium on Applied Algebra, Algebraic Algorithms, and Error-Correcting Codes (pp. 130-139). Berlin, Heidelberg: Springer Berlin Heidelberg.
3. Gel'fand, I. M., Zelevinskii, A. V., & Kapranov, M. M. (1989). Hypergeometric functions and toral manifolds. Functional Analysis and its applications, 23(2), 94-106.
4. Sturmfels, B. (1991). Grobner bases of toric varieties. Tohoku Mathematical Journal, Second Series, 43(2), 249-261.
5. Diaconis, P., & Sturmfels, B. (1998). Algebraic algorithms for sampling from conditional distributions. The Annals of statistics, 26(1), 363-397.

6. Maclagan, D., Thomas, R. R., Faridi, S., Gold, L., Jayanthan, A. V., Khetan, A., & Puthenpurakal, T. (2005). Computational algebra and combinatorics of toric ideals. Commutative algebra and combinatorics, 4.