ICMRS

2022

5TH INTERNATIONAL CONFERENCE ON MATHEMATICAL AND RELATED SCIENCES

ANTALYA, TURKEY
OCTOBER 27-30, 2022

5TH INTERNATIONAL CONFERENCE ON MATHEMATICAL AND RELATED SCIENCES

BOOK OF PROCEEDINGS

ISBN: 978-605-70978-5-9

2022

5^{TH} INTERNATIONAL CONFERENCE ON MATHEMATICAL AND RELATED SCIENCES

ICMRS 2022

ANTALYA-TURKEY

OCTOBER 27-30, 2022

<u>EDITORS</u>

ERHAN SET, PhD

AHMET OCAK AKDEMİR, PhD

MEHMET ZEKİ SARIKAYA, PHD

ALPER EKİNCİ, PhD

All papers have been peer reviewed.

CHAIR

Prof. Dr. Ahmet Ocak AKDEMİR
Ağrı İbrahim Çeçen University *Turkey*

ORGANIZING COMMITTEE

Prof. Dr. Erhan SET
Ordu University *Turkey*

Prof. Dr. Mehmet Zeki SARIKAYA
Düzce University *Turkey*

Assoc. Prof. Dr. Alper EKİNCİ
Bandırma Onyedi Eylül University *Turkey*

SECRETARY

PhD Stud. Barış ÇELİK
Ordu University *Turkey*

KEYNOTE SPEAKERS

Prof. Dr. Maria Alessandra RAGUSA
University of Catania – Italy

Prof. Dr. Thabet ABDELJAWAD
Prince Sultan University, Saudi Arabia

Prof. Dr. Miodrag SPALEVIC
University of Belgrade-SERBIA

Prof. Dr. Zlatko PAVIC
University of Slavonski Brod-CROATIA

Prof. Dr. Özkan ÖCALAN
Akdeniz University-TURKEY

SCIENTIFIC COMMITTEE

Abdallah BENAISSA
University of Batna Algeria

Abdullah ÇAĞMAN
Ağrı İbrahim Çeçen University Turkey

Abdullah ERGÜN
Sivas Cumhuriyet University Turkey

Ali AKGÜL
Siirt University Turkey

Ali ALLAHVERDI
Kuwait University Kuwait

Ali ÇAKMAK
Bitlis Eren University Turkey

Ali EMROUZNEJOD
Aston University U.K.

Allal GUESSAB
Université de Pau et des Pays de I'Adour France

Anthony SOFO
Victoria University Australia

Arif RAFIQ
Hajvery University Pakistan
İmdat İŞCAN Giresun University Turkey

İrem BAĞLAN
Kocaeli University Turkey

İsa YILDIRIM
Atatürk University Turkey

Ömür DEVECİ
Kafkas University Turkey

Çetin YILDIZ
Atatürk University Turkey

Bayram ŞAHİN
Ege University Turkey

Binod Chandra TRIPATHY
Tripura University India

Borhan ALBISS
Jordan University of Science and Technology Jordan

Burhaneddin İZGİ
İstanbul Technical University Turkey

Coşkun ÇETİN
California State University USA

Djamila SEBA
University of Boumerdes Algeria

Donal O REGAN
National University of Ireland Ireland

Ekrem KADIOĞLU
Atatürk University Turkey

Elnara MAMMADOVA
Baku State University Azerbaijan

Elvan AKIN
Missouri University of Science and Technology USA

Emrah Evren KARA
Düzce University Turkey

Ercan ÇELİK
Atatürk University Turkey

Erdal ÜNLÜYOL
Ordu University Turkey

Erdinç DÜNDAR
Afyon Kocatepe University Turkey

Erhan DENİZ
Kafkas University Turkey

Evrim GÜVEN
Kocaeli University Turkey

Fahd JARAD
Çankaya University Turkey

Farooq Ahmad GUJAR
Bahauddin Zakariya University Pakistan

Feng QI
Henan Polytechnic University China

Ferhan M. ATICI
Western Kentucky University USA

Fikret ALIEV
Baku State University Azerbaijan

Fuad KITTANEH
University of Jordan Amman Jordan

Furkan YILDIRIM
Atatürk University Turkey

Gabil YAGUB
Kafkas University Turkey

Gabriela CRISTESCU
Aurel Vlaicu University of Arad Romania

Gamar H. MAMMADOVA
Baku State University Azerbaijan

Günay ÖZTÜRK
İzmir Democracy University Turkey

Halit ORHAN
Atatürk University Turkey

Hamdullah ŞEVLİ
Van Yüzüncü Yıl University Turkey

Hasan FURKAN
Kahramanmaraş Sütçü İmam University Turkey

Havva KAVURMACI-ÖNALAN
Yüzüncü Yıl University Turkey

József SANDOR
Babeş-Bolyai University Romania

Jehad AL JARADEN
Al Hussein Bin Talal University Jordan

Juan Luis García GUIRAO
Technical University of Cartagena Spain

Junesang CHOI
Dongguk University Republic of Korea

Kürşat AKBULUT
Ataturk University Turkey

Khaled NAWAFLEH
Mutah University Jordan

Khuram Ali KHAN
University of Sargodha Pakistan

Ljubisa KOCINAC
University of Nis Serbia

Luca GRILLI
Università di Foggia Italy

M. Emin ÖZDEMİR
Bursa Uludağ University Turkey

Mahmoud ALREFAEI
Jordan University of Science and Technology Jordan

Manaf MANAFLI
Adıyaman University Turkey

Marcela V. MIHAI
University of Craiova Romania

Maria Alessandra RAGUSA
University of Catania Italy

Mehmet Eyüp KİRİŞ
Afyon Kocatepe University Turkey

Mehmet KORKMAZ
Ordu University Turkey

Merve AVCI-ARDIÇ
Adıyaman University Turkey

Michal FECKAN
Comenius University Slovakia

Mikail ET
Fırat University Turkey

Mohammad Uzair AWAN
Goverment College University Pakistan

Muhammad Adil KHAN
University of Peshawar Pakistan

Muhammed Al REFAI
United Arab Emirates University UAE

Murat ÖZDEMİR
Atatürk University Turkey

Murat ÇAĞLAR
Erzurum Technical University Turkey

Murat BEŞENK
Pamukkale University Turkey

Mustafa Ali DOKUYUCU
Ağrı İbrahim Çeçen University Turkey

Mustafa GÜRBÜZ
Ağrı İbrahim Çeçen University Turkey

Mustafa Kemal YILDIZ
Afyon Kocatepe University Turkey

Nargiz SAFAROVA
Baku State University Azerbaijan

Nazir AHMAD MIR
Mir Preston University Pakistan

Necip ŞİMŞEK
İstanbul Commerce University Turkey

Nesip AKTAN
Necmettin Erbakan University Turkey

Nidal ANAKIRA
Irbid National University Jordan

Nigar YILDIRIM AKSOY
Kafkas University Turkey

Nizami MUSTAFA
Kafkas University Turkey

Praveen AGARWAL
Anand International College of
Engineering India

Qamrul Hasan ANSARI
Aligarh Muslim University India

Ram U. VERMA
The University of Akron USA

Saad Ihsan BUTT
COMSATS
Institute of Information Technology
Pakistan

Saburou SAITOH
University of Aveiro Portugal

Sait TAŞ
Atatürk University Turkey

Samer Al GHOUR
Jordan University of Science and
Technology Jordan

Sanja VAROSANEC
University of Zagreb Croatia

Süleyman ŞENYURT
Ordu University Turkey

Seher ASLANCI
Alanya Alaaddin Keykubat University
Turkey

Selahattin MADEN
Ordu University Turkey

Sever Silvestru DRAGOMIR
Victoria University Australia

Shigeyoshi OWA
Kinki University Japan

Syed Abdul MOHUIDDINE
King Abdulaziz University Saudi Arabia

Taha Yasin ÖZTÜRK
Kafkas University Turkey

Tahsin ÖNER
Ege University Turkey

Tamer UĞUR
Atatürk University Turkey

Thabet ABDELJAWAD
Prince Sultan University Saudi Arabia

Themistocles M. RASSIAS
National Technical University of Athens
Greece

Vasile BERINDE
North University Baia Mare Romania

Yanlin LI
Hangzhou Normal University China

Yıldıray ÇELİK
Ordu University Turkey

Zlatko PAVIC
University of Slavonski Brod Croatia

Zoubir DAHMANI
University of Mostaganem Algeria

CONTENTS

On Generalization of Some Integral Inequalities with the Help of AB-Fractional Integral Operators and *s*-Convex Functions

Erhan SET[1] and Ali KARAOĞLAN[1]

[1]*Department of Mathematics, Faculty of Sciences and Arts, Ordu University, Ordu, TURKEY*

erhanset@yahoo.com
alikaraoglan@odu.edu.tr

ABSTRACT

One of the known methods in the literature to obtain different versions, generalizations and extensions of inequalities is to use different classes of convex functions such that s-convexity, m-convexity, harmonically convexity, r-convexity, quasi-convexity et al. Also, in recent years, fractional integral operators have become a frequently used method to obtain new versions, generalizations and extensions of classical integral inequalities. One of these operators is AB-fractional integral operator defined by Atangana and Baleanu. In this study, we use the AB-fractional integral operators to establish some new generalized integral inequalities that are connected with the celebrated Hermite Hadamard integral inequality with the help of s-convex functions in the second sense.

1. INTRODUCTION

Convex functions, which are of high importance for the theory of inequality in terms of wide range of applications and features, are the focus of researchers in many applied fields such as convex programming. Let's start by remembering this useful function class.

Definition 1.1 *The function* $f:[a,b]\subseteq \mathbb{R}\to\mathbb{R}$, *is said to be convex if the following inequality holds*

$$f(\lambda x+(1-\lambda)y)\le \lambda f(x)+(1-\lambda)f(y) \quad (1.1)$$

for all $x,y\in[a,b]$ and $\lambda\in[0,1]$. We say that f is concave if $(-f)$ is convex.

Definition 1.2 *(see [3],[6]) Let* $0<s\le 1$. *A function* $f:[0,\infty)\to\mathbb{R}$, *is said to be s-Breckner convex or s-convex in the second sense, if for every* $x,y\in[0,\infty)$ *and* $\alpha,\beta\ge 0$ *with* $\alpha+\beta=1$, *we have*

$$f(\alpha x+\beta y)\le \alpha^s f(x)+\beta^s f(y). \tag{1.2}$$

The set of all s-convex functions in the second sense is denoted by K_s^2.

It can be easily seen that for $s=1$, s-convexity reduces to the ordinary convexity of functions defined on $[0,\infty)$.

Hermite-Hadamard inequality, one of the first types of inequality in which convex functions are used in inequality theory, is an aesthetic inequality whose lower and upper bounds can be expressed in arithmetic mean. This famous inequality is expressed as follows.

Assume that $f: I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex mapping defined on the interval I of $\mathbb{R}$ where $a < b$. The following statement;

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.3}$$

holds and known as Hermite-Hadamard inequality.

In [5] Dragomir and Fitzpatrick proved a variant of Hadamard¡¦s inequality which holds for s-convex functions in the second sense.

Theorem 1.1 *Suppose that* $f:[0,\infty) \to \mathbb{R}$ *is an s-convex function in the second sense, where* $s \in (0,1)$, *and let* $a,b \in [0,\infty)$, $a<b$. *If* $f \in L[a,b]$, *then the following inequalities hold*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{s+1}.$$

Definition 1.3 *(see, [4],[9]) Let* $H^1(a,b)$ *be the Sobolev space of order 1 given as follows*

$$H^1(a,b) = \{u \in L_2(a,b) : u' \in L_2(a,b)\}.$$

In this paper, we will denote normalization function as $B(\alpha)$ with $B(0) = B(1) = 1$ and $\Gamma(.)$ is Gamma function.

Left hand side of Atangana-Baleanu integral operator has been defined as follows.

Definition 1.4 *[2] The fractional integral associate to the new fractional derivative with non-local kernel of a function* $f \in H^1(a,b)$ *as defined:*

$${}_{a}^{AB}I^{\alpha}\{f(t)\} = \frac{1-\alpha}{B(\alpha)} f(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_a^t f(y)(t-y)^{\alpha-1}dy$$

where $b > a, \alpha \in (0,1]$.

In [1], the authors have given the right hand side of integral operator as following;

$${}^{AB}I_b^{\alpha}\{f(t)\} = \frac{1-\alpha}{B(\alpha)} f(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_t^b f(y)(y-t)^{\alpha-1}dy.$$

The Gamma function $\Gamma(z)$ developed by Euler is usually defined as follow.

Definition 1.5 *[8] Assume that* $\Re(z) > 0$, *the Gamma function is denoted by* $\Gamma(z)$ *and defined as follow.*

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt.$$

Definition 1.6 *[8] Assume that* $\Re(\eta) > 0$ *and* $\Re(\rho) > 0$, *the Beta function is denoted by* $\beta(\eta,\rho)$ *and defined as*

$$\beta(\eta,\rho) = \int_0^1 t^{\eta-1}(1-t)^{\rho-1} dt.$$

The main aim of this study is to obtain new Hermite-Hadamard type inequalities via Atangana-Baleanu integral operators for s-convex functions in the second sense using identity that provided by Set et al. in [7]. The main findings are supported by some reduced results.

2. MAIN RESULTS

We will give the identity that provided by Set et al. to obtain main results as follow.

Lemma 2.1 *[7] Let* $f:[a,b]\to\mathbb{R}$ *be differentiable function on* (a,b) *with* $a<b$. *Then we have the following identity for Atangana-Baleanu fractional integral operators*

$$\begin{aligned}
&{}^{AB}_{a}I^{\alpha}\{f(t)\} + {}^{AB}I_b^{\alpha}\{f(t)\} - \frac{(t-a)^{\alpha} f(a) + (b-t)^{\alpha} f(b)}{B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha) f(t)}{B(\alpha)} \\
&= \frac{(t-a)^{\alpha+1}}{(\alpha+1)B(\alpha)\Gamma(\alpha)} f'(a) + \frac{(t-a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)} \int_0^1 (1-k)^{\alpha+1} f''\big(kt+(1-k)a\big)dk \\
&- \frac{(b-t)^{\alpha+1}}{(\alpha+1)B(\alpha)\Gamma(\alpha)} f'(b) + \frac{(b-t)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)} \int_0^1 k^{\alpha+1} f''\big(kb+(1-k)t\big)dk
\end{aligned}$$

where $\alpha\in(0,1], t\in[a,b]$, $B(\alpha)$ is normalization function and $\Gamma(.)$ is Gamma function.

Theorem 2.1 *Let* $f:[a,b]\to\mathbb{R}$ *be differentiable function on* (a,b) *with* $0\le a<b$ *and* $f''\in L_1[a,b]$. *If* $|f''|$ *is a* s*-convex function in the second sense, we have the following inequality for Atangana-Baleanu fractional integral operators*

$$\begin{aligned}
&\left| {}^{AB}_{a}I^{\alpha}\{f(t)\} + {}^{AB}I_b^{\alpha}\{f(t)\} - \frac{(t-a)^{\alpha} f(a) + (b-t)^{\alpha} f(b)}{B(\alpha)\Gamma(\alpha)} \right. \\
&\left. - \frac{(t-a)^{\alpha+1} f'(a) - (b-t)^{\alpha+1} f'(b)}{(\alpha+1)B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha) f(t)}{B(\alpha)} \right| \\
&\le \frac{(t-a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)} \left[\beta(s+1,\alpha+2)\left|f''(t)\right| + \frac{\left|f''(a)\right|}{(\alpha+s+2)} \right]
\end{aligned} \tag{2.1}$$

$$+\frac{(b-t)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left[\beta(s+1,\alpha+2)\left|f''(t)\right|+\frac{\left|f''(b)\right|}{(\alpha+s+2)}\right]$$

where $t\in[a,b]$, $\alpha\in(0,1]$, $s\in(0,1]$, $B(\alpha)$ is normalization function, $\Gamma(.)$ is Gamma function and $\beta(.)$ is Beta function.

Proof. By using the identity that is given in Lemma 2.1, we can write

$$\begin{aligned}
&|{}^{AB}_{a}I^{\alpha}\{f(t)\}+{}^{AB}I^{\alpha}_{b}\{f(t)\}-\frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)}\\
&-\frac{(t-a)^{\alpha+1}f'(a)-(b-t)^{\alpha+1}f'(b)}{(\alpha+1)B(\alpha)\Gamma(\alpha)}-\frac{2(1-\alpha)f(t)}{B(\alpha)}|\\
&=|\frac{(t-a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\int_0^1(1-k)^{\alpha+1}f''\left(kt+(1-k)a\right)dk\\
&+\frac{(b-t)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\int_0^1k^{\alpha+1}f''\left(kb+(1-k)t\right)dk|\\
&\le\frac{(t-a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\int_0^1(1-k)^{\alpha+1}|f''\left(kt+(1-k)a\right)|dk\\
&+\frac{(b-t)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\int_0^1k^{\alpha+1}|f''\left(kb+(1-k)t\right)|dk.
\end{aligned}$$

By using s-convexity in the second sense of $|f''|$, we get

$$\begin{aligned}
&|{}^{AB}_{a}I^{\alpha}\{f(t)\}+{}^{AB}I^{\alpha}_{b}\{f(t)\}-\frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)}\\
&-\frac{(t-a)^{\alpha+1}f'(a)-(b-t)^{\alpha+1}f'(b)}{(\alpha+1)B(\alpha)\Gamma(\alpha)}-\frac{2(1-\alpha)f(t)}{B(\alpha)}|\\
&\le\frac{(t-a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\int_0^1(1-k)^{\alpha+1}|f''\left(kt+(1-k)a\right)|dk\\
&+\frac{(b-t)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\int_0^1k^{\alpha+1}|f''\left(kb+(1-k)t\right)|dk\\
&\le\frac{(t-a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\int_0^1(1-k)^{\alpha+1}[k^s\left|f''(t)\right|+(1-k)^s\left|f''(a)\right|]dk\\
&+\frac{(b-t)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\int_0^1k^{\alpha+1}[k^s\left|f''(b)\right|+(1-k)^s\left|f''(t)\right|]dk\\
&=\frac{(t-a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left[\beta(s+1,\alpha+2)\left|f''(t)\right|+\frac{\left|f''(a)\right|}{(\alpha+s+2)}\right]
\end{aligned}$$

$$+\frac{(b-t)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left[\beta(s+1,\alpha+2)\left|f^{''}(t)\right|+\frac{\left|f^{''}(b)\right|}{(\alpha+s+2)}\right].$$

So, the proof is completed.

Remark 2.1 *In Theorem 2.1, if we choose* $s=1$, *the inequality (2.1) reduces to the inequality in Theorem 2.1 in [7].*

Theorem 2.2 *Let* $f:[a,b]\to\mathbb{R}$ *be differentiable function on* (a,b) *with* $0\le a<b$ *and* $f^{''}\in L_1[a,b]$. *If* $|f^{''}|^q$ *is a* s*-convex function in the second sense, we have the following inequality for Atangana-Baleanu fractional integral operators*

$$\begin{aligned}&\left|{}_{a}^{AB}I^{\alpha}\{f(t)\}+{}^{AB}I_{b}^{\alpha}\{f(t)\}-\frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)}\right.\\&\left.-\frac{(t-a)^{\alpha+1}f^{'}(a)-(b-t)^{\alpha+1}f^{'}(b)}{(\alpha+1)B(\alpha)\Gamma(\alpha)}-\frac{2(1-\alpha)f(t)}{B(\alpha)}\right|\\&\le\frac{(t-a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha p+p+1}\right)^{\frac{1}{p}}\left(\frac{\left|f^{''}(a)\right|^{q}+\left|f^{''}(t)\right|^{q}}{s+1}\right)^{\frac{1}{q}}\\&+\frac{(b-t)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha p+p+1}\right)^{\frac{1}{p}}\left(\frac{\left|f^{''}(t)\right|^{q}+\left|f^{''}(b)\right|^{q}}{s+1}\right)^{\frac{1}{q}}\end{aligned}\tag{2.2}$$

where $p^{-1}+q^{-1}=1$, $q>1$, $t\in[a,b]$, $s\in(0,1]$, $\alpha\in(0,1]$, $B(\alpha)$ is normalization function, $\Gamma(.)$ is Gamma function and $\beta(.)$ is Beta function.

Proof. By using Lemma 2.1, we have

$$\begin{aligned}&\left|{}_{a}^{AB}I^{\alpha}\{f(t)\}+{}^{AB}I_{b}^{\alpha}\{f(t)\}-\frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)}\right.\\&\left.-\frac{(t-a)^{\alpha+1}f^{'}(a)-(b-t)^{\alpha+1}f^{'}(b)}{(\alpha+1)B(\alpha)\Gamma(\alpha)}-\frac{2(1-\alpha)f(t)}{B(\alpha)}\right|\\&\le\frac{(t-a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\int_0^1(1-k)^{\alpha+1}\left|f^{''}(kt+(1-k)a)\right|dk\\&+\frac{(b-t)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\int_0^1k^{\alpha+1}\left|f^{''}(kb+(1-k)t)\right|dk.\end{aligned}$$

By applying Hölder inequality, we get

$$\begin{aligned}
&\Big| {}^{AB}_{a}I^{\alpha}\{f(t)\}+{}^{AB}I^{\alpha}_{b}\{f(t)\}-\frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)}\\
&-\frac{(t-a)^{\alpha+1}f'(a)-(b-t)^{\alpha+1}f'(b)}{(\alpha+1)B(\alpha)\Gamma(\alpha)}-\frac{2(1-\alpha)f(t)}{B(\alpha)}\Big|\\
&\leq\frac{(t-a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1(1-k)^{(\alpha+1)p}dk\right)^{\frac{1}{p}}\left(\int_0^1\left|f''(kt+(1-k)a)\right|^q dk\right)^{\frac{1}{q}}\right]\\
&+\frac{(b-t)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1 k^{(\alpha+1)p}dk\right)^{\frac{1}{p}}\left(\int_0^1\left|f''(kb+(1-k)t)\right|^q dk\right)^{\frac{1}{q}}\right].
\end{aligned}$$

By using s-convexity in the second sense of $|f''|^q$, we obtain

$$\begin{aligned}
&\Big| {}^{AB}_{a}I^{\alpha}\{f(t)\}+{}^{AB}I^{\alpha}_{b}\{f(t)\}-\frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)}\\
&-\frac{(t-a)^{\alpha+1}f'(a)-(b-t)^{\alpha+1}f'(b)}{(\alpha+1)B(\alpha)\Gamma(\alpha)}-\frac{2(1-\alpha)f(t)}{B(\alpha)}\Big|\\
&\leq\frac{(t-a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1(1-k)^{(\alpha+1)p}dk\right)^{\frac{1}{p}}\left(\int_0^1[k^s\left|f''(t)\right|^q+(1-k)^s\left|f''(a)\right|^q]dk\right)^{\frac{1}{q}}\right]\\
&+\frac{(b-t)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left[\left(\int_0^1 k^{(\alpha+1)p}dk\right)^{\frac{1}{p}}\left(\int_0^1[k^s\left|f''(b)\right|^q+(1-k)^s\left|f''(t)\right|^q]dk\right)^{\frac{1}{q}}\right]\\
&=\frac{(t-a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha p+p+1}\right)^{\frac{1}{p}}\left(\frac{\left|f''(a)\right|^q+\left|f''(t)\right|^q}{s+1}\right)^{\frac{1}{q}}\\
&+\frac{(b-t)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(\frac{1}{\alpha p+p+1}\right)^{\frac{1}{p}}\left(\frac{\left|f''(t)\right|^q+\left|f''(b)\right|^q}{s+1}\right)^{\frac{1}{q}}.
\end{aligned}$$

So, the proof is completed.

Remark 2.2 *In Theorem 2.2, if we choose $s=1$, the inequality (2.2) reduces to the inequality in Theorem 2.2 in [7].*

Theorem 2.3 *Let $f:[a,b]\to\mathbb{R}$ be differentiable function on (a,b) with $0\leq a<b$ and $f''\in L_1[a,b]$. If $|f''|^q$ is a s-convex function in the second sense, we have the following inequality for Atangana-Baleanu fractional integral operators*

$$\begin{aligned}
&\left| {}^{AB}_{a}I^{\alpha}\{f(t)\} + {}^{AB}I^{\alpha}_{b}\{f(t)\} - \frac{(t-a)^{\alpha} f(a) + (b-t)^{\alpha} f(b)}{B(\alpha)\Gamma(\alpha)} \right. \\
&\left. - \frac{(t-a)^{\alpha+1} f'(a) - (b-t)^{\alpha+1} f'(b)}{(\alpha+1)B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha) f(t)}{B(\alpha)} \right| \\
&\leq \frac{(t-a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)} \left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}} \left(\beta(s+1,\alpha+2)\left|f''(t)\right|^{q} + \frac{\left|f''(a)\right|^{q}}{\alpha+s+2} \right)^{\frac{1}{q}} \\
&+ \frac{(b-t)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)} \left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}} \left(\beta(s+1,\alpha+2)\left|f''(t)\right|^{q} + \frac{\left|f''(b)\right|^{q}}{\alpha+s+2} \right)^{\frac{1}{q}}
\end{aligned} \tag{2.3}$$

where $t \in [a,b]$, $s \in (0,1]$, $\alpha \in (0,1]$, $q \geq 1$, $B(\alpha)$ is normalization function, $\Gamma(.)$ is Gamma function and $\beta(.)$ is Beta function.

Proof. By using Lemma 2.1, we get

$$\begin{aligned}
&\left| {}^{AB}_{a}I^{\alpha}\{f(t)\} + {}^{AB}I^{\alpha}_{b}\{f(t)\} - \frac{(t-a)^{\alpha} f(a) + (b-t)^{\alpha} f(b)}{B(\alpha)\Gamma(\alpha)} \right. \\
&\left. - \frac{(t-a)^{\alpha+1} f'(a) - (b-t)^{\alpha+1} f'(b)}{(\alpha+1)B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha) f(t)}{B(\alpha)} \right| \\
&\leq \frac{(t-a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)} \int_0^1 (1-k)^{\alpha+1} \left| f''\left(kt+(1-k)a\right)\right| dk \\
&+ \frac{(b-t)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)} \int_0^1 k^{\alpha+1} \left| f''\left(kb+(1-k)t\right)\right| dk.
\end{aligned}$$

By applying power mean inequality, we get

$$\begin{aligned}
&\left| {}^{AB}_{a}I^{\alpha}\{f(t)\} + {}^{AB}I^{\alpha}_{b}\{f(t)\} - \frac{(t-a)^{\alpha} f(a) + (b-t)^{\alpha} f(b)}{B(\alpha)\Gamma(\alpha)} \right. \\
&\left. - \frac{(t-a)^{\alpha+1} f'(a) - (b-t)^{\alpha+1} f'(b)}{(\alpha+1)B(\alpha)\Gamma(\alpha)} - \frac{2(1-\alpha) f(t)}{B(\alpha)} \right| \\
&\leq \frac{(t-a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)} \left[\left(\int_0^1 (1-k)^{\alpha+1} dk \right)^{1-\frac{1}{q}} \left(\int_0^1 (1-k)^{\alpha+1} \left| f''\left(kt+(1-k)a\right)\right|^{q} dk \right)^{\frac{1}{q}} \right] \\
&+ \frac{(b-t)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)} \left[\left(\int_0^1 k^{\alpha+1} dk \right)^{1-\frac{1}{q}} \left(\int_0^1 k^{\alpha+1} \left| f''\left(kb+(1-k)t\right)\right|^{q} dk \right)^{\frac{1}{q}} \right].
\end{aligned}$$

By using s-convexity in the second sense of $|f''|^q$, we get

$$
\begin{aligned}
&|{}_{a}^{AB}I^{\alpha}\{f(t)\}+{}^{AB}I_{b}^{\alpha}\{f(t)\}-\frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)}\\
&-\frac{(t-a)^{\alpha+1}f'(a)-(b-t)^{\alpha+1}f'(b)}{(\alpha+1)B(\alpha)\Gamma(\alpha)}-\frac{2(1-\alpha)f(t)}{B(\alpha)}|\\
&\leq\frac{(t-a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}[\left(\int_0^1(1-k)^{\alpha+1}dk\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1(1-k)^{\alpha+1}[k^s\left|f''(t)\right|^q+(1-k)^s\left|f''(a)\right|^q]dk\right)^{\frac{1}{q}}]\\
&+\frac{(b-t)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}[\left(\int_0^1k^{\alpha+1}dk\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1k^{\alpha+1}[k^s\left|f''(b)\right|^q+(1-k)^s\left|f''(t)\right|^q]dk\right)^{\frac{1}{q}}]\\
&=\frac{(t-a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left[\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}}\left(\beta(s+1,\alpha+2)\left|f''(t)\right|^q+\frac{\left|f''(a)\right|^q}{\alpha+s+2}\right)^{\frac{1}{q}}\right]\\
&+\frac{(b-t)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left[\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}}\left(\frac{\left|f''(b)\right|^q}{\alpha+s+2}+\beta(s+1,\alpha+2)\left|f''(t)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

So, the proof is completed.

Remark 2.3 *In Theorem 2.3, if we choose* $s=1$, *the inequality (2.3) reduces to the inequality in Theorem 2.4 in [7].*

Theorem 2.4 *Let* $f:[a,b]\to\mathbb{R}$ *be differentiable function on* (a,b) *with* $0\leq a<b$ *and* $f''\in L_1[a,b]$. *If* $|f''|^q$ *is a* s*-convex function in the second sense, we have the following inequality for Atangana-Baleanu fractional integral operators*

$$
\begin{aligned}
&|{}_{a}^{AB}I^{\alpha}\{f(t)\}+{}^{AB}I_{b}^{\alpha}\{f(t)\}-\frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)}\\
&-\frac{(t-a)^{\alpha+1}f'(a)-(b-t)^{\alpha+1}f'(b)}{(\alpha+1)B(\alpha)\Gamma(\alpha)}-\frac{2(1-\alpha)f(t)}{B(\alpha)}|
\end{aligned}
\tag{2.4}
$$

$$\leq \frac{(t-a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(\frac{1}{p(\alpha p+p+1)}+\frac{\left|f^{''}(a)\right|^q+\left|f^{''}(t)\right|^q}{q(s+1)}\right)$$
$$+\frac{(b-t)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(\frac{1}{p(\alpha p+p+1)}+\frac{\left|f^{''}(t)\right|^q+\left|f^{''}(b)\right|^q}{q(s+1)}\right)$$

where $t\in[a,b]$, $s\in(0,1]$, $\alpha\in(0,1]$, $p^{-1}+q^{-1}=1$, $q>1$, $B(\alpha)$ is normalization function, $\Gamma(.)$ is Gamma function and $\beta(.)$ is Beta function.

Proof. By using Lemma 2.1, we get

$$\left| {}_a^{AB}I^{\alpha}\{f(t)\}+{}^{AB}I_b^{\alpha}\{f(t)\}-\frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)}\right.$$
$$\left.-\frac{(t-a)^{\alpha+1}f^{'}(a)-(b-t)^{\alpha+1}f^{'}(b)}{(\alpha+1)B(\alpha)\Gamma(\alpha)}-\frac{2(1-\alpha)f(t)}{B(\alpha)}\right|$$
$$\leq \frac{(t-a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\int_0^1(1-k)^{\alpha+1}\left|f^{''}\left(kt+(1-k)a\right)\right|dk$$
$$+\frac{(b-t)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\int_0^1 k^{\alpha+1}\left|f^{''}\left(kb+(1-k)t\right)\right|dk.$$

By using the Young inequality as $xy\leq\frac{1}{p}x^p+\frac{1}{q}y^q$, we have

$$\left| {}_a^{AB}I^{\alpha}\{f(t)\}+{}^{AB}I_b^{\alpha}\{f(t)\}-\frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)}\right.$$
$$\left.-\frac{(t-a)^{\alpha+1}f^{'}(a)-(b-t)^{\alpha+1}f^{'}(b)}{(\alpha+1)B(\alpha)\Gamma(\alpha)}-\frac{2(1-\alpha)f(t)}{B(\alpha)}\right|$$
$$\leq \frac{(t-a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left[\frac{1}{p}\int_0^1(1-k)^{(\alpha+1)p}dk+\frac{1}{q}\int_0^1\left|f^{''}\left(kt+(1-k)a\right)\right|^q dk\right]$$
$$+\frac{(b-t)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left[\frac{1}{p}\int_0^1 k^{(\alpha+1)p}dk+\frac{1}{q}\int_0^1\left|f^{''}\left(kb+(1-k)t\right)\right|^q dk\right].$$

By using s-convexity in the second sense of $|f^{''}|^q$, we have

$$\left| {}_a^{AB}I^{\alpha}\{f(t)\}+{}^{AB}I_b^{\alpha}\{f(t)\}-\frac{(t-a)^{\alpha}f(a)+(b-t)^{\alpha}f(b)}{B(\alpha)\Gamma(\alpha)}\right.$$

$$-\frac{(t-a)^{\alpha+1} f^{'}(a)-(b-t)^{\alpha+1} f^{'}(b)}{(\alpha+1)B(\alpha)\Gamma(\alpha)}-\frac{2(1-\alpha) f(t)}{B(\alpha)}|$$
$$\leq \frac{(t-a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left[\frac{1}{p}\int_0^1 (1-k)^{(\alpha+1)p} dk+\frac{1}{q}\int_0^1 [k^s\left|f^{''}(t)\right|^q+(1-k)^s\left|f^{''}(a)\right|^q]dk\right]$$
$$+\frac{(b-t)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left[\frac{1}{p}\int_0^1 k^{(\alpha+1)p} dk+\frac{1}{q}\int_0^1 [k^s\left|f^{''}(b)\right|^q+(1-k)^s\left|f^{''}(t)\right|^q]dk\right]$$
$$=\frac{(t-a)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(\frac{1}{p(\alpha p+p+1)}+\frac{\left|f^{''}(a)\right|^q+\left|f^{''}(t)\right|^q}{q(s+1)}\right)$$
$$+\frac{(b-t)^{\alpha+2}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(\frac{1}{p(\alpha p+p+1)}+\frac{\left|f^{''}(t)\right|^q+\left|f^{''}(b)\right|^q}{q(s+1)}\right).$$

So, the proof is completed.

Remark 2.4 *In Theorem 2.4, if we choose* $s=1$, *the inequality (2.4) reduces to the inequality in Theorem 2.3 in [7].*

REFERENCES

1. T. Abdeljawad and D. Baleanu, Integration by parts and its applications of a new nonlocal fractional derivative with Mittag-Leffler nonsingular kernel, J. Nonlinear Sci. Appl., 10 (2017), 1098-1107.
2. A. Atangana and D. Baleanu, New fractional derivatices with non-local and non-singular kernel, Theory and Application to Heat Transfer Model, Thermal Science, Year 2016, Vol. 20, No. 2, pp. 763-769
3. WW. Breckner, Stetigkeitsaussagenf ureine Klass ever all gemeinerter konvexer funktionen in topologisc henlianeren Raumen, Pupl. İnst. Math., 23 (1978), 13-20.
4. M. Caputo and M. Fabrizio, A new definition of fractional derivative without singular kernel, Progress in Fractional Differentiation and Applications, 1(2) (2015), 73-85.
5. S.S. Dragomir and S. Fitzpatrick, The Hadamard's inequality for s-convex functions in the second sense. Demonstr. Math. 32(4), 687-696 (1999)
6. H. Hudzik and L. Maligranda, Some remarks on s-convex functions, Aequationes Math., 48 (1994), 100-111.
7. E. Set, S. I. Butt, A. O. Akdemir, A. Karaoğan, New integral inequalities for differentiable convex functions via Atangana-Baleanu fractional integral operators, Chaos, Solitons and Fractals, 143 (2021) 110554.
8. H. M. Srivastava, J. Choi, Zeta and q-Zeta Functions and Associated Series and Integrals, Elsevier (2012).
9. G. Tınaztepe, I. Yeşilce Işık, S. Kemali, G. Adilov, Certain Inequalities for s-Convex Functions via Caputo Fractional Derivative and Caputo-Fabrizio Integral Operator, Journal of Mathematical Extension, 15 (23) (2021), 1-18.

On New Versions of Bullen-type Inequalities Based on Conformable Fractional Integrals

Hasan KARA[1] , Fatih HEZENCİ[1] , Hüseyin BUDAK[1] and Mustafa Kemal YILDIZ[2]

[1]*Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce-TURKEY*
[2]*Department of Mathematics, Faculty of Science and Arts, Afyon Kocatepe University, Afyonkarahisar-TURKEY*

hasan64kara@gmail.com
fatihezenci@gmail.com
hsyn.budak@gmail.com
myildiz@aku.edu.tr

ABSTRACT

This research is on the new versions of Bullen-type inequalities. These inequalities established by means of convex mappings include conformable fractional integral operators. Obtaining these inequalities, well-known Hölder inequality and power mean inequality are also utilized. The resulting Bullen-type inequalities are a generalization of some of the studies on this subject, including Riemann integrals and Riemann-Liouville integrals. What's more, new results are obtained through special choices.

1. INTRODUCTION

Convex theory is a subject that has been used in many fields of optimization theory, energy systems, engineering applications, and physics and has guided many studies in the literature. Also, the convex theory is an available way to solve many problems from different branches of mathematics. Convexity theory has an important place in these branches of mathematics, especially in inequalities. Hermite-Hadamard, Simpson, Newton, and Bullen-type inequalities are the most well-known of these inequalities.

Today's researchers use the derivative and integral as a tool to produce different solutions to almost all of the problems that arise in each of the fields of basic science such as mathematics, physics, chemistry, and engineering such as industry and electricity. Classical derivative, classical integral, and differential concepts although it solves most of the problems that arise in many areas of technology, these concepts are insufficient in solving many of them. Fractional calculus has been the solution to these problems. Many authors began to deal with the discrete versions of this fractional calculus benefiting from the theory of time scales. Two basic approaches are used to do this fractional calculation. The first approach called the Riemann-Liouville approach, in addition to repeating the integral operator n times, he made it possible to convert it to an integral with the Cauchy formula where then $n!$ is changed to the

Gamma function, and hence the fractional integral operator of non-integer order is described. These integers were then used to find the Riemann-Liouville and Caputo fractional derivatives. The second approach is the Grünwald–Letnikov approach which is the aid of iterating the derivative n times and then fractionalizing involving the Gamma function in the binomial coefficients. In the results obtained with these approaches, the calculations seemed complicated as the product rule and the chain rule properties were lost from the properties of the derivative. That's why the Conformable fractional approach was developed, which depends on the fundamental definition of the derivative in [18]. In [2], the author proved that the conformable approach in [18] cannot yield good results when compared to the Caputo definition for specific functions. This flaw in the conformable definition was avoided by some extensions of the conformable approach [12, 22]. Based on these approaches, Jarad obtained the definitions of conformable fractional integrals in [15]. Inspired by all these studies, fractional calculus attracts researchers every day.

In [3], Bullen introduced Bullen-type inequalities in 1978, which is named after him, and which has guided many studies in the literature. Dragomir and Wang acquired a natural generalization of Bullen's inequality in [8]. Sarikaya et al. acquired generalized Bullen-type inequalities in [20]. Erden and Sarikaya proved the generalized inequalities of Bullen-type with the aid of the local fractional integrals on fractal sets in [10]. Du et al. utilized the generalized fractional integrals to discover Bullen-type inequalities in [9]. Hwang et al. have constructed some new Hermite-Hadamard-type, Bullen-type, and Simpson-type inequalities in [11] with the aid of fractional integrals. Starting from the equality they obtained, $\dot{I}s$can et al. found some new Hermite-Hadamard and Bullen-type inequalities via functions whose derivatives in modulus at certain power are convex in [13]. Tseng et al. acquired some Hadamard-type and Bullen-type inequalities via Lipschitzian functions and give several applications with help of the special means in [21]. With help of the some Euler-type identities, Matic et al. presented a generalization of Bullen-Simpson's inequality based on (2ô)-convex mappings in [19]. Çakmak presented some new Bullen-type inequalities based on differentiable mappings with the help of the *s*–convexity and Riemann-Liouville fractional integral operators via Gauss hyper-geometric function in [4]. Also the author proved a new identity based on differentiable mappings and established some new inequalities via differentiable mappings with the aid of the h-convex mappings involving Bullen-type inequalities in [5]. In [16], Kara et al. obtained the above and below bounds via parameterized-type inequalities utilizing the Riemann–Liouville fractional integral operators and limited second derivative functions. These presented some new Bullen-type inequalities according to the specific choices of the parameter. Besides all this, Çakmak has done two different studies in [6] and [7] on Bullen-type inequalities involving a different conformable fractional integral operator.

With the help of the continuing research and mentioned papers above, we will acquire some Bullen-type inequalities via differentiable convex mappings involving conformable fractional integral operators. The entire form of study takes the form of four sections including the introduction. In Section 2, the fundamentals definitions of Riemann-Liouville integral operators and conformable integrals will be explained for building our main results. In addition, recalls will be made about gamma, beta, and incompleted beta functions, which are well-known in the literature. In Section 3, an identity will be present for the case of differentiable convex mappings involving the conformable fractional integral operators. By utilizing this equality, we prove some Bullen-type inequalities via convex mappings with the help of conformable fractional integrals. More precisely, Hölder and power-mean inequalities, which are well-known in the literature, will use in some of the proven inequalities. Furthermore, we also present some corollaries and remarks. Finally, in Section 4, ideas that will guide the researchers will be given. Interested researchers will be informed that new versions of the inequalities we have acquired can be derived via different fractional integrals.

2. PRELIMINARIES

In order to create our main results, in this section, the gamma function, beta function, incomplete beta function, the definition of Rieman-Liouville integrals, and the definition of Conformable fractional integrals will be presented.

Definition 1. *The gamma function, beta function, and incomplete beta function are defined by*

$$\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}dt,$$

$$B(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}dt,$$

and

$$\boldsymbol{B}(x,y,r) := \int_0^r t^{x-1}(1-t)^{y-1}dt,$$

respectively. Here, $0 < x, y < \infty$ and $r \in [0,1]$.

Riemann-Liouville integral operators are defined by as follows

Definition 2. [17] *For $f \in L_1[a,b]$, the Riemann-Liouville integrals of order $\beta > 0$ are given by*

$$J_{a+}^{\beta}f(x) = \frac{1}{\Gamma(\beta)}\int_a^x (x-t)^{\beta-1} f(t)dt, \quad x > a \tag{2.1}$$

and

$$J_{b-}^{\beta}f(x) = \frac{1}{\Gamma(\beta)}\int_x^b (t-x)^{\beta-1} f(t)dt, \quad x < b. \tag{2.2}$$

The Riemann-Liouville integrals will be equal to classical integrals for the condition $\beta = 1$.

In [15], Jarad et al. gave the fractional conformable integral operators.

Definition 3. [15] *For $f \in L_1[a,b]$, the fractional conformable integral operator ${}^{\beta}J_{a+}^{\alpha}f(x)$ and $\beta J_{b-}^{\alpha}f(x)$ of order $\beta \in C$, $Re(\beta) > 0$ and $\alpha \in (0,1]$ are presented by*

$${}^{\beta}J_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(\beta)}\int_a^x \left(\frac{(x-a)^{\alpha}-(t-a)^{\alpha}}{\alpha}\right)^{\beta-1} \frac{f(t)}{(t-a)^{1-\alpha}}dt, \qquad t > a \tag{2.3}$$

and

$$^{\beta}J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\beta)}\int_x^b \left(\frac{(b-x)^{\alpha}-(b-t)^{\alpha}}{\alpha}\right)^{\beta-1} \frac{f(t)}{(b-t)^{1-\alpha}}dt, \qquad t < b, \tag{2.4}$$

respectively.

If we consider $\alpha = 1$, then the fractional integral in (2.3) reduces to the Riemann-Liouville fractional integral in (2.1). Furthermore, the fractional integral in (2.4) coincides with the Riemann-Liouville fractional integral in (2.2) when $\alpha = 1$. For some recent results connected with fractional integral inequalities, see [1, 14] and the references cited therein.

3. MAIN RESULTS

In this section, we use conformable fractional integrals to construct Bullen-type inequalities for differentiable convex mappings. First, let's set up the following identity to establish Bullen-type inequalities.

Lemma 1. *Consider that* $f: [a, b] \to R$ *is a differentiable mapping on* (a, b) *such that* $f' \in L_1[a, b]$. *Then, the following equality holds:*

$$\begin{aligned}&\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right] \\ &-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left[\ {}^{\beta}J^{\alpha}_{b-}f\left(\frac{a+b}{2}\right)+\ {}^{\beta}J^{\alpha}_{a+}f\left(\frac{a+b}{2}\right)\right] \\ &=\frac{(b-a)\alpha^{\beta}}{4}\int_0^1\left[\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right] \\ &\times\left[f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)-f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right]dt.\end{aligned} \tag{3.1}$$

Proof. Employing the integration by parts gives,

$$I_1 = \int_0^1\left[\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right]\left[f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)-f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right]dt \tag{3.2}$$

$$=\frac{2}{b-a}\left[\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right]f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\Bigg|_0^1$$

$$-\frac{2\beta}{b-a}\int_0^1\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1}f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt$$

$$+\frac{2}{b-a}\left[\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right]f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\Bigg|_0^1$$

$$-\frac{2\beta}{b-a}\int_0^1\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1}f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt$$

$$=\frac{2}{(b-a)\alpha^{\beta}}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]$$

$$-\frac{2\beta}{b-a}\int_0^1\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1}f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt$$

$$-\frac{2\beta}{b-a}\int_0^1\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1}f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt.$$

With the help of the change of variables in (3.2), then the equality (3.2) turns into the following equality

$$I_1=\frac{2}{(b-a)\alpha^\beta}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right] \tag{3.3}$$

$$-\left(\frac{2}{b-a}\right)^{\alpha\beta+1}\frac{\Gamma(\beta+1)}{\Gamma(\beta)}\int_{\frac{a+b}{2}}^{b}\left(\frac{\left(\frac{b-a}{2}\right)^\alpha-(b-x)^\alpha}{\alpha}\right)^{\beta-1}\frac{f(x)}{(b-x)^{1-\alpha}}f(x)d\,x$$

$$-\left(\frac{2}{b-a}\right)^{\alpha\beta+1}\frac{\Gamma(\beta+1)}{\Gamma(\beta)}\int_{a}^{\frac{a+b}{2}}\left(\frac{\left(\frac{b-a}{2}\right)^\alpha-(x-a)^\alpha}{\alpha}\right)^{\beta-1}\frac{f(x)}{(x-a)^{1-\alpha}}f(x)dx$$

$$=\frac{2}{(b-a)\alpha^\beta}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]$$

$$-\left(\frac{2}{b-a}\right)^{\alpha\beta+1}\Gamma(\beta+1)\left[{}^{\beta}J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{\beta}J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)\right].$$

If the expression (3.3) is multiplied by $\frac{(b-a)\alpha^\beta}{4}$, then the proof of Lemma 1 becomes clear.

Theorem 1. *Suppose that $f:[a,b]\to R$ is a differentiable mapping on (a,b) such that $f'\in L_1[a,b]$ and $|f'|$ is convex on $[a,b]$. Under these conditions, the following inequality is derived:*

$$\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{2^{\alpha\beta-1}\alpha^\beta\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left[{}^{\beta}J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{\beta}J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right| \tag{3.4}$$

$$\le\frac{(b-a)\alpha^\beta}{4}\phi_1(\alpha,\beta)[|f'(b)|+|f'(a)|].$$

Here,

$$\phi_1(\alpha,\beta)=\int_0^1\left|\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^\beta-\frac{1}{2\alpha^\beta}\right|dt=\frac{1}{\alpha^\beta}\left[\frac{1}{2}-\left(1-\left(\frac{1}{2}\right)^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}}\right] \tag{3.5}$$

$$+\frac{1}{\alpha^{\beta+1}}\left[B\left(\frac{1}{\alpha},\beta+1\right)-2\boldsymbol{B}\left(\frac{1}{\alpha},\beta+1,\left(\frac{1}{2}\right)^{\frac{1}{\beta}}\right)\right],$$

where beta function and incomplete beta function are denoted as B and $\boldsymbol{B}$, respectively.

Proof. If we take the absolute value of both sides of (3.1), then we obtain

$$\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]\right. \tag{3.6}$$

$$\left.-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left[\ {}^{\beta}J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)+\ {}^{\beta}J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq\frac{(b-a)\alpha^{\beta}}{4}\int_0^1\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right|\left|f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|dt$$

$$+\frac{(b-a)\alpha^{\beta}}{4}\int_0^1\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt.$$

It is known that $|f'|$ is convex on $[a,b]$. It follows

$$\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left[\ {}^{\beta}J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{\beta}J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq\frac{(b-a)\alpha^{\beta}}{4}\int_0^1\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right|\left(\frac{1-t}{2}|f'(b)|+\frac{1+t}{2}|f'(a)|+\frac{1-t}{2}|f'(a)|+\frac{1+t}{2}|f'(b)|\right)dt$$

$$=\frac{(b-a)\alpha^{\beta}}{4}\int_0^1\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right|[|f'(b)|+|f'(a)|]dt.$$

Thus, the proof of Theorem 1 is finished.

Corollary 1. *If we set $\alpha=1$ in Theorem 1, then we have*

$$\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^{\beta}}\left[J_{b-}^{\beta}f\left(\frac{a+b}{2}\right)+J_{a+}^{\beta}f\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq\frac{b-a}{4}\phi_1(1,\beta)[|f'(b)|+|f'(a)|],$$

where

$$\phi_1(1,\beta)=\int_0^1\left|t^{\beta}-\frac{1}{2}\right|dt=\frac{\beta}{\beta+1}\left(\frac{1}{2}\right)^{\frac{1}{\beta}}+\frac{1}{\beta+1}-\frac{1}{2}. \tag{3.7}$$

Remark 1. *Let us consider $\alpha=1$ and $\beta=1$ in Theorem 1, then we acquire*

$$\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{b-a}{16}[|f'(b)|+|f'(a)|],$$

which is given by Hwang et al. in [11, Remark 4.2].

Theorem 2. *Suppose that $f:[a,b]\to R$ is a differentiable mapping on (a,b), such that $f'\in L_1[a,b]$. In addition, suppose that $|f'|^q$ is convex on $[a,b]$ with $q>1$. Then the following inequality can be written*

$$\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]\right. \tag{3.8}$$

$$\left.-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left[\,{}^{\beta}J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)+\,{}^{\beta}J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq\frac{(b-a)\alpha^{\beta}}{4}\left(\psi_1^{\alpha,\beta}(p)\right)^{\frac{1}{p}}\left[\left(\frac{|f'(b)|^q+3|f'(a)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right]$$

$$\leq\frac{(b-a)\alpha^{\beta}}{4}\left(4\psi_1^{\alpha,\beta}(p)\right)^{\frac{1}{p}}[|f'(a)|+|f'(b)|].$$

Here, $\frac{1}{p}+\frac{1}{q}=1$ and

$$\psi_1^{\alpha,\beta}(p)=\int_0^1\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right|^p dt.$$

Proof. If the properties of Hölder's inequality are used in (3.6), then we acquire

$$\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left[\,{}^{\beta}J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{\beta}J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq\frac{(b-a)\alpha^{\beta}}{4}\left\{\left(\int_0^1\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right\}.$$

Applying the convexity of $|f'|^q$ on $[a,b]$, we have the following inequality

$$\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left[\,{}^{\beta}J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{\beta}J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq\frac{(b-a)\alpha^{\beta}}{4}\left\{\left(\int_0^1\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right|^p dt\right)^{\frac{1}{p}}\right.$$

$$\left.\times\left[\left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right]\right\}$$

$$= \frac{(b-a)\alpha^{\beta}}{4}\left(\psi_1^{\alpha,\beta}(p)\right)^{\frac{1}{p}}\left[\left(\frac{|f'(b)|^q+3|f'(a)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right].$$

The second inequality of Theorem 2 can be acquired immediately by letting $\varpi_1 = 3|f''(a)|^q$, $\rho_1 = |f''(b)|^q$, $\varpi_2 = |f''(a)|^q$ and $\rho_2 = 3|f''(b)|^q$ and applying the inequality:

$$\sum_{k=1}^{n}(\varpi_k+\rho_k)^s \leq \sum_{k=1}^{n}\varpi_k^s + \sum_{k=1}^{n}\rho_k^s, \quad 0 \leq s < 1.$$

Thus, the proof of Theorem 2 is completed.

Corollary 2. *If Theorem 2 is evaluated as $\alpha = 1$, the following result is obtained*

$$\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^{\beta}}\left[J_{b-}^{\beta}f\left(\frac{a+b}{2}\right)+J_{a+}^{\beta}f\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq \frac{b-a}{4}\left(\Xi^{\beta}(p)\right)^{\frac{1}{p}}\left[\left(\frac{|f'(b)|^q+3|f'(a)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{b-a}{4}\left(4\Xi^{\beta}(p)\right)^{\frac{1}{p}}[|f'(a)|+|f'(b)|].$$

Here,

$$\Xi^{\beta}(p) = \int_0^1\left|t^{\beta}-\frac{1}{2}\right|^p dt.$$

Corollary 3. *When we consider $\alpha = 1$ and $\beta = 1$, we can write the Theorem 2 in the following format*

$$\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{b-a}{8}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|f'(b)|^q+3|f'(a)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{b-a}{8}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}[|f'(a)|+|f'(b)|].$$

Theorem 3. *Consider the existence of a differentiable mapping such that $f:[a,b]\to R$ on (a,b) and $f' \in L_1[a,b]$. Let's also assume that the function $|f'|^q$ is convex on $[a,b]$ with $q \geq 1$. Then, the following inequality is established:*

$$\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left[{}^{\beta}J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{\beta}J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq \frac{(b-a)\alpha^{\beta}}{4}\left(\phi_1(\alpha,\beta)\right)^{1-\frac{1}{q}}\left[\left(\frac{(\phi_1(\alpha,\beta)+\phi_2(\alpha,\beta))}{2}|f'(b)|^q+\frac{(\phi_1(\alpha,\beta)-\phi_2(\alpha,\beta))}{2}|f'(a)|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{(\phi_1(\alpha,\beta)-\phi_2(\alpha,\beta))}{2}|f'(b)|^q+\frac{(\phi_1(\alpha,\beta)+\phi_2(\alpha,\beta))}{2}|f'(a)|^q\right)^{\frac{1}{q}}\right].$$

Here, $\phi_1(\alpha,\beta)$ is described as in (3.5) and

$$\phi_2(\alpha,\beta)=\int_0^1 t\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right|dt=\frac{1}{\alpha^{\beta}}\left\{\frac{1}{2}-\left[1-\left(1-\left(\frac{1}{2}\right)^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}}\right]^2-\frac{1}{4}\right\}$$

$$+\frac{1}{\alpha^{\beta+1}}\left[B\left(\frac{1}{\alpha},\beta+1\right)-2\boldsymbol{B}\left(\frac{1}{\alpha},\beta+1,\left(\frac{1}{2}\right)^{\frac{1}{\beta}}\right)\right.$$

$$\left.-B\left(\frac{2}{\alpha},\beta+1\right)+2\boldsymbol{B}\left(\frac{2}{\alpha},\beta+1,\left(\frac{1}{2}\right)^{\frac{1}{\beta}}\right)\right],$$

where B and $\boldsymbol{B}$ denote the beta function and incomplete beta function, respectively.

Proof. With the help of the power-mean inequality, we have

$$\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left[{}^{\beta}J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{\beta}J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq\frac{(b-a)\alpha^{\beta}}{4}\left\{\left(\int_0^1\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right|dt\right)^{1-\frac{1}{q}}\right.$$

$$\times\left[\left(\int_0^1\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right|\left|f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$\left.\left.+\left(\int_0^1\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right]\right\}.$$

Since $|f'|^q$ is convex on $[a,b]$, we establish

$$\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left[\,{}^{\beta}J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{\beta}J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq\frac{(b-a)\alpha^{\beta}}{4}\left\{\left(\int_0^1\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right|dt\right)^{1-\frac{1}{q}}\right.$$

$$\times\left[\left(\int_0^1\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right|\left(\frac{1+t}{2}|f'(b)|^q+\frac{1-t}{2}|f'(a)|^q\right)dt\right)^{\frac{1}{q}}\right.$$

$$\left.\left.+\left(\int_0^1\left|\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right|\left(\frac{1+t}{2}|f'(a)|^q+\frac{1-t}{2}|f'(b)|^q\right)dt\right)^{\frac{1}{q}}\right]\right\}.$$

With this calculation, the proof ends.

Corollary 4. *If $\alpha=1$ in Theorem 3, the following inequality is obtained*

$$\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^{\beta}}\left[J_{b-}^{\beta}f\left(\frac{a+b}{2}\right)+J_{a+}^{\beta}f\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq\frac{(b-a)}{4}\left(\phi_1(1,\beta)\right)^{1-\frac{1}{q}}\left[\left(\frac{\Omega_1(\beta)\left|f'(b)\right|^q+\Omega_2(\beta)\left|f'(a)\right|^q}{2}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{\Omega_1(\beta)\left|f'(a)\right|^q+\Omega_2(\beta)\left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}\right],$$

where $\phi_1(1,\beta)$ is defined as in (3.7) and

$$\phi_2(1,\beta)=\frac{\beta-2}{2(\beta+2)}\left(\frac{1}{2}\right)^{\frac{2}{\beta}}+\frac{1}{\beta+2}-\frac{1}{4}.$$

Here,

$$\Omega_1(\beta)=\phi_1(1,\beta)+\phi_2(1,\beta)$$

$$=\left(\frac{\beta}{\beta+1}\right)\left(\frac{1}{2}\right)^{1/\beta}+\left(\frac{\beta-2}{2(\beta+2)}\right)\left(\frac{1}{2}\right)^{\frac{2}{\beta}}+\frac{2\beta+3}{(\beta+1)(\beta+2)}-\frac{3}{4}$$

and

$$\Omega_2(\beta)=\phi_1(1,\beta)-\phi_2(1,\beta))$$

$$=\left(\frac{\beta}{\beta+1}\right)\left(\frac{1}{2}\right)^{1/\beta}-\left(\frac{\beta-2}{2(\beta+2)}\right)\left(\frac{1}{2}\right)^{\frac{2}{\beta}}+\frac{1}{(\beta+1)(\beta+2)}-\frac{1}{4}.$$

Corollary 5. *If we take $\alpha=1$ and $\beta=1$ in Theorem 3, we acquire*

$$\left|\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{b-a}{16}\left[\left(\frac{7|f'(b)|^q+5|f'(a)|^q}{12}\right)^{\frac{1}{q}}+\left(\frac{5|f'(b)|^q+7|f'(a)|^q}{12}\right)^{\frac{1}{q}}\right].$$

4. CONCLUSION

In the current research, we derive the new Bullen-type inequalities by making use of Conformable fractional integrals. Convexity of the function, Hölder and power-mean inequalities are used in these inequalities. Furthermore, special choices of the variables in the theorems, generalizations of some articles, and new results were found. In the future, the authors may derive new inequalities of different fractional types related to these Bullen-type inequalities. Interested readers can also establish new inequalities using different kinds of convexities. These inequalities created are new as far as we know and according to the literature review. These inequalities will inspire new studies in various fields of mathematics.

REFERENCES

[1] T. Abdeljawad, *On conformable fractional calculus*, J. Comput. Appl. Math. 279, 2015, 57-66.
[2] A. A. Abdelhakim, *The flaw in the conformable calculus: It is conformable because it is not fractional*, Fractional Calculus and Appied Analysis, 22, 2019, 242-254.
[3] P. S. Bullen, *Error estimates for some elementary quadrature rules*, Publikacije Elektrotehničkog fakulteta. Serija Matematika i fizika, (602/633), 1978, 97-103.
[4] M. Çakmak, M. *Refinements of Bullen-type inequalities for s–convex functions via Riemann-Liouville fractional integrals involving Gauss hypergeometric function*, Journal of Interdisciplinary Mathematics, 22(6), 2019, 975-989.
[5] M. Çakmak, *The differentiable h -convex functions involving the Bullen inequality*, Acta Universitatis Apulensis, (65), 2021, 29.
[6] M. Çakmak, *On some Bullen-type inequalities via conformable fractional integrals*, Journal of Scientific Perspectives, 3(4), 2019, 285-298.
[7] M. Çakmak, *Some Bullen-type inequalities for conformable fractional integrals*, General Mathematics, 2020, 28(2), 3–17.
[8] S. S. Dragomir, and S. O. N. G. Wang, *A generalization of Bullen's inequality for convex mappings and its applications*, Soochow Journal of Mathematics, 24(2), 1998, 97-103.
[9] T. Du, C. Luo and Z. Cao, *On the Bullen-type inequalities via generalized fractional integrals and their applications*, Fractals, Vol. 29, No. 07, 2021.
[10] S. Erden, M. Z. Sarikaya, *Generalized Bullen-type inequalities for local fractional integrals and its applications*, Palestine Journal of Mathematics, 2020, 9(2).
[11] H. R. Hwang, K. L. Tseng, and K. C. Hsu, New inequalities for fractional integrals and their applications, *Turkish Journal of Mathematics*, 40(3), (2016) 471–486.
[12] A. Hyder, A.H. Soliman, *A new generalized θ -conformable calculus and its applications in mathematical physics*, Physica Scripta, 96, 2020, 015208.

[13] I. İscan, T. Toplu and F. Yetgin, *Some new inequalities on generalization of Hermite-Hadamard and Bullen type inequalities, applications to trapezoidal and midpoint formula*, Kragujevac Journal of Mathematics, 45(4), 2021, 647-657.
[14] F. Jarad, T. Abdeljawad, D. Baleanu, *On the generalized fractional derivatives and their Caputo modification*, J. Nonlinear Sci. Appl. 10(5), 2017, 2607-2619.
[15] F. Jarad, E. Uğurlu, T. Abdeljawad, and D. Baleanu, *On a new class of fractional operators*, Advances in Difference Equations, (2017), 2017, 247.
[16] H. Kara, H. Budak, and F. Hezenci, *New extensions of the parameterized inequalities based on Riemann–Liouville fractional integrals*. Mathematics, 10(18), 2022, 3374.
[17] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204, Elsevier Sci. B.V., Amsterdam, 2006.
[18] R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh, *A new definition of fractional derivative*, Journal of Computational and Applied Mathematics, 264, 2014, 65-70.
[19] M. Matić, J. Pečarić and A. Vukelić, *On generalization of Bullen-Simpson's inequality*, The Rocky Mountain Journal of Mathematics, 2005, 1727-1754.
[20] M. Z. Sarikaya, H. Budak, *Some integral inequalities for local fractional integrals*, International Journal of Analysis and Applications, 14(1), 2017, 9-19
[21] K. L. Tseng, S. R. Hwang, and K. C. Hsu, *Hadamard-type and Bullen-type inequalities for Lipschitzian functions and their applications*, Computers & Mathematics with Applications, 64(4), 2012, 651-660.
[22] D. Zhao, M. Luo, *General conformable fractional derivative and its physical interpretation*, Calcolo, 54, 2017, 903-917.

Midpoint Inequalities for Superquadratic Functions

Mehmet Zeki SARIKAYA and Necmettin ALP

Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce-TURKEY

sarikayamz@gmail.com

ABSTRACT

The main object of this paper is to present the Hermite-Hadamard inequalities for superquadratics functions. We establish the midpoint inequalities with using a important integral identity (see Lemma 4) for differentiable superquadratic mappings.

1. INTRODUCTION

The usefulness of inequalities involving convex functions is realized from the very beginning and is now widely acknowledged as one of the prime driving forces behind the development of several modern branches of mathematics and has been given considerable attention. Some famous results for such estimations consist of Hermite-Hadamard, trapezoid, midpoint, Simpson or Jensen inequalities ect.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following double inequality is well known in the literature as the Hermite-Hadamard inequalities [9]:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}. \qquad (1.1)$$

The most well-known inequalities related to the integral mean of a convex function are the Hermite Hadamard inequalities. It gives an estimate from both sides of the mean value of a convex function and also ensure the integrability of convex function. It is also a matter of great interest and one has to note that some of the classical inequalities for means can be obtained from Hadamard's inequality under the utility of peculiar convex functions f : These inequalities for convex functions play a crucial role in analysis and as well as in other areas of pure and applied mathematics. The absolute value of the difference of the second part of the (1.1) inequalities is known as the trapezoidal inequality in the literature and was given by Dragomir and Agarwal in 1998 [8]. Then, in 2004, the absolute value of the difference of the first part of the (1.1) inequalities, known as the midpoint inequality by Kirmanci, was given [10].

Recall that a convex function satisfies

$$\phi(y)-\phi(x)\geq C(x)(y-x)$$

for all x, y where $C(x)=\phi'(x)$ (or, if ϕ is not differentiable at x, any number between the left and right derivatives at x). In [1], the authors introduced the class of superquadratic functions, defined as follows. Superquadratic functions have been introduced as a modification of convex functions in [1]. The definition of the superquadratic functions on which the study will be built is as follows:

Definition 1. *[1] A function $\phi : [0,\infty)\to R$ is superquadratic provided that for all $x\geq 0$ there exists a constant $C_x \in R$ such that*

$$\phi(y)\geq \phi(x)+C_x(y-x)+\phi(|y-x|) \qquad (1.2)$$

for all $y\geq 0$.

We say that ϕ is subquadratic if $-\phi$ is a superquadratic function. It is shown that if ϕ is a nonnegative superquadratic function, then ϕ is convex and $\phi(0)=\phi'(0)=0$.

Remark 1. *For $\phi(x)=x^2$, equality holds in (1.2), with $C(x)=2x$. Also, the definition, with $y=x$, forces $\phi(0)\leq 0$, from which it follows that one can always take $C(0)$ to be 0. If ϕ is differentiable and satisfies $\phi(0)=\phi'(0)=0$, then one sees easily that the $C(x)$ appearing in the definition is necessarily $\phi'(x)$.*

Some basic properties and examples of superquadratic functions can be found in [1].

Lemma 1. *[1] Let ϕ be a superquadratic function with C_x as in Definition Definition Superquadratic.*

(i) Then $\phi(0)\leq 0$.

(ii) If $\phi(0)=\phi'(0)=0$, then $C_x=\phi'(x)$ whenever ϕ is differentiable at $x>0$.

(iii) If $\phi\geq 0$, then ϕ is convex and $\phi(0)=\phi'(0)=0$.

Lemma 2. *[2] Suppose that ϕ is superquadratic and non-negative. Then ϕ is convex and increasing. Also, if $C(x)$ is as in (1.2), then $C(x)\geq 0$.*

Proof. Convexity is shown in [2]. Together with $\phi(0)=0$ and $\phi(x)\geq 0$, this implies that ϕ is increasing. As mentioned already, we can take $C(0)=0$. For $x>0$ and $y<x$, we can rewrite (1.2) as

$$C(x)\geq \frac{\phi(x)-\phi(y)+\phi(x-y)}{x-y}\geq 0.$$

The next lemma (essentially Lemma 3.2 of [1]) gives a simple sufficient condition. We include a sketch of the proof for completeness. The next result gives a sufficient condition when convexity(concavity) implies super(sub)quaradicity.

Lemma 3. *[1] If ϕ' is convex(concave) and $\phi(0)=\phi'(0)=0$, then ϕ is super(sub)quadratic. The converse of is not true.*

Proof. First, since ϕ' is convex and $\phi'(0)=0$, we have $\phi'(x)\le[x/(x+y)]\phi'(x+y)$ for $x,y\ge 0$, and hence

$$\phi'(x)+\phi'(y)\le\phi'(x+y)$$

(that is, ϕ' is superadditive). Now let $y>x\ge 0$. Then

$$\phi(y)-\phi(x)-\phi(y-x)-(y-x)\phi'(x)$$
$$=\int_0^{y-x}[\phi'(t+x)-\phi'(t)-\phi'(x)]dt\ge 0.$$

Similarly for the case $x>y\ge 0$.

Remark 2. *Hence $\phi(x)=x^p$ is superquadratic for $p\ge 2$ and subquadratic for $1<p\le 2$. (It is also easily seen that $\phi(x)=x^p$ is subquadratic for $0<p\le 1$, with $C(x)=0$). Subquadraticity does always not imply concavity; i.e.,there exists a subquadratic function which is convex. For example, $\phi(x)=x^p$, $x\ge 0$ and $1\le p\le 2$ is subquadratic and convex.*

The following inequality is due to M. Petrovic [11].

Theorem 1. *Let $0<a<\infty$, and let $f:[0,a)\to R$ be a continuous and convex function. Then for every $n\in N$ and every $x_1,x_2,...,x_n\in[0,a)$ such that $x_1+x_2+,...+x_n\in[0,a)$ we have*

$$f(x_1)+f(x_2)+f(x_3)+...+f(x_n)\le f(x_1+x_2+,...+x_n)+(n-1)f(0).$$

Banic and Varosanec in [6] gave an important result with characterizations of the superquadratic functions, which are analogous to the well known characterizations of the convex functions: For the function $\phi:[0,\infty)\to R$ the following conditions are equivalent:

A) ϕ is a superquadratic function, i.e., there exists a constant C_x such that

$$\phi(y)\ge\phi(x)+C_x(y-x)+\phi(|y-x|),\qquad \forall x,y>0.$$

B) The inequality

$$\phi(tx_1+(1-t)x_2)\le t\phi(x_1)+(1-t)\phi(x_2)-t\phi((1-t)|x_2-x_1|)-(1-t)\phi(t|x_2-x_1|)\quad(1.3)$$

holds for all $x_1,x_2\ge 0$ and $t\in[0,1]$.

C) For all $x_1, x_2 \geq 0$ and $x_1 \leq x \leq x_2$ we have

$$\frac{\phi(x_1)-\phi(x)-\phi(x-x_1)}{x_1-x} \leq \frac{\phi(x_2)-\phi(x)-\phi(x-x_2)}{x_2-x}.$$

In 2008, Banic et al. [7] proved the following Hermite-Hadamard inequality for superquadratic functions by using Jensen's inequality for superquadratic functions:

Theorem 2. *Let* $\phi : [0,\infty) \to \mathbb{R}$ *be an integrable superquadratic function and* $0 \leq a \leq b$ *. Then*

$$\begin{aligned}&\phi\left(\frac{a+b}{2}\right)+\frac{1}{b-a}\int_a^b \phi\left(\left|x-\frac{a+b}{2}\right|\right)dx \\ &\leq \frac{1}{b-a}\int_a^b \phi(x)\,dx \qquad (1.4) \\ &\leq \frac{\phi(a)+\phi(b)}{2}-\frac{1}{(b-a)^2}\int_a^b \{(b-x)\phi(x-a)+(x-a)\phi(b-x)\}\,dx.\end{aligned}$$

In this paper, we firstly prove the Hermite-Hadamard inequalities using the definition of the superquadratic function in (1.3). Then, we will investigate some inequalities connected with the left part of the inequality (1.4). In order to achieve our goals, we have to establish a important integral identity (see Lemma 4) for differentiable superquadratic mappings.

2. MIDPOINT INEQUALITIES

Now, let's start our main results with a new proof of the above Theorem 2:

Proof. For $t \in [0,1]$, let $x = ta + (1-t)b$, $y = (1-t)a + tb$. Since ϕ is a superquadratic function, then

$$\phi\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\phi(ta+(1-t)b)+\frac{1}{2}\phi((1-t)a+tb)-\phi\left(\frac{b-a}{2}|1-2t|\right). \quad (2.1)$$

Then integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned}&\phi\left(\frac{a+b}{2}\right) \\ &\leq \frac{1}{2}\int_0^1 \phi(ta+(1-t)b)\,dt+\frac{1}{2}\int_0^1 \phi((1-t)a+tb)\,dt \\ &-\int_0^1 \phi\left(\frac{b-a}{2}|1-2t|\right)dt.\end{aligned}$$

With change of variable $x = ta + (1-t)b,\ y = (1-t)a + tb$ and $1 - 2t = x - \frac{a+b}{2}$ in the above integrals, we obtain

$$\phi\left(\frac{a+b}{2}\right)$$

$$\leq \frac{1}{b-a}\int_a^b \phi(x)dx - \frac{1}{(b-a)^2}\int_a^b \phi\left(\left|x - \frac{a+b}{2}\right|\right)dx$$

and the first inequality is proved.

To prove the other half of the inequality in (1.4), since ϕ is a superquadratic function, for every $t \in [0,1]$, we have,

$$\phi(ta + (1-t)b) + \phi((1-t)a + tb)$$

$$\leq f(a) + f(b) - 2t\phi((1-t)|b-a|) - 2(1-t)\phi(t|b-a|).$$

Then integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\int_0^1 \phi(ta + (1-t)b)dt + \int_0^1 \phi((1-t)a + tb)dt$$

$$\leq f(a) + f(b) - 2\int_0^1 t\phi((1-t)|b-a|)dt - 2\int_0^1 (1-t)\phi(t|b-a|)dt.$$

With change of variable $x = ta + (1-t)b,\ y = (1-t)a + tb$, $1 - t = \frac{x-a}{b-a}$ and $t = \frac{b-x}{b-a}$ in the above integrals, we get

$$\frac{1}{b-a}\int_a^b \phi(x)dx \leq \frac{f(a) + f(b)}{2} - \frac{1}{(b-a)^2}\int_a^b \{(b-x)\phi(x-a) + (x-a)\phi(b-x)\}dx$$

and the second inequality is proved.

Lemma 4. *Let* $\phi : [0,\infty) \to \mathbb{R}$ *be an integrable superquadratic function and* $0 \leq a \leq b$*, then the following equality holds:*

$$(b-a)\int_0^{\frac{1}{2}} t\left\{\phi'(bt + (1-t)a) + \phi'\left(\frac{a+b}{2} - (bt + (1-t)a)\right)\right\}dt$$
$$+(b-a)\int_{\frac{1}{2}}^1 (t-1)\left\{\phi'(bt + (1-t)a) - \phi'\left(bt + (1-t)a - \frac{a+b}{2}\right)\right\}dt + \phi(0) \quad (2.2)$$
$$= \phi\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b \left\{\phi(x) - \phi\left(\left|x - \frac{a+b}{2}\right|\right)\right\}dx.$$

Proof. It's easy to calculate the following equalities

$$
\begin{aligned}
I_1 &= \int_0^{\frac{1}{2}} t\left\{\phi'(bt+(1-t)a)+\phi'\left(\frac{a+b}{2}-(bt+(1-t)a)\right)\right\}dt \\
&= \left[\frac{1}{b-a}t\left\{\phi(bt+(1-t)a)-\phi\left(\frac{a+b}{2}-(bt+(1-t)a)\right)\right\}\right]_0^{\frac{1}{2}} \\
&\quad -\frac{1}{b-a}\int_0^{\frac{1}{2}}\left\{\phi(bt+(1-t)a)-\phi\left(\frac{a+b}{2}-(bt+(1-t)a)\right)\right\}dt \\
&= \frac{1}{2(b-a)}\left\{\phi\left(\frac{a+b}{2}\right)-\phi(0)\right\} \\
&\quad -\frac{1}{b-a}\int_0^{\frac{1}{2}}\left\{\phi(bt+(1-t)a)-\phi\left(\frac{a+b}{2}-(bt+(1-t)a)\right)\right\}dt,
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 &= \int_{\frac{1}{2}}^{1}(t-1)\left\{\phi'(bt+(1-t)a)-\phi'\left(bt+(1-t)a-\frac{a+b}{2}\right)\right\}dt \\
&= \left[\frac{1}{b-a}(t-1)\left\{\phi(bt+(1-t)a)-\phi\left(bt+(1-t)a-\frac{a+b}{2}\right)\right\}\right]_{\frac{1}{2}}^{1} \\
&\quad -\frac{1}{b-a}\int_{\frac{1}{2}}^{1}\left\{\phi(bt+(1-t)a)-\phi\left(bt+(1-t)a-\frac{a+b}{2}\right)\right\}dt \\
&= \frac{1}{2(b-a)}\left\{\phi\left(\frac{a+b}{2}\right)-\phi(0)\right\} \\
&\quad -\frac{1}{b-a}\int_{\frac{1}{2}}^{1}\left\{\phi(bt+(1-t)a)-\phi\left(bt+(1-t)a-\frac{a+b}{2}\right)\right\}dt.
\end{aligned}
$$

If we add I_1 and I_2 and multiply by $(b-a)$, the proof is completed as below:

$$
\begin{aligned}
&(b-a)(I_1+I_2) \\
&= \left[\phi\left(\frac{a+b}{2}\right)-\phi(0)\right]-\int_0^1\left\{\phi(bt+(1-t)a)-\phi\left(\left|(bt+(1-t)a)-\frac{a+b}{2}\right|\right)\right\}dt \\
&= \left[\phi\left(\frac{a+b}{2}\right)-\phi(0)\right]-\frac{1}{b-a}\int_a^b\left\{\phi(x)-\phi\left(\left|x-\frac{a+b}{2}\right|\right)\right\}dx.
\end{aligned}
$$

Theorem 3. *Let* $\phi:[0,\infty)\to\mathbb{R}$ *be an integrable superquadratic function and* $0\le a\le b$ *and* $|\phi'|$ *be a superquadratic. Then, the following midpoint inequality holds:*

$$\left|\phi\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b\left\{\phi(x)-\phi\left(\left|x-\frac{a+b}{2}\right|\right)\right\}dx\right|$$

$$\leq\frac{(b-a)}{4}\left\{\frac{|\phi'(a)|+2|\phi'(b)|}{3}+|\phi'(0)|\right\}+\phi(0).$$

Proof. Taking absulate value of (2.2), we get

$$\begin{aligned}&\left|\phi\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b\left\{\phi(x)-\phi\left(\left|x-\frac{a+b}{2}\right|\right)\right\}dx\right|\\&\leq(b-a)\int_0^{\frac{1}{2}}t\left\{|\phi'(bt+(1-t)a)|+\left|\phi'\left(\frac{a+b}{2}-(bt+(1-t)a)\right)\right|\right\}dt\\&+(b-a)\int_{\frac{1}{2}}^1(1-t)|\phi'(bt+(1-t)a)|dt\\&+(b-a)\int_{\frac{1}{2}}^1(1-t)\left|\phi'\left(bt+(1-t)a-\frac{a+b}{2}\right)\right|dt+\phi(0).\end{aligned}\tag{2.3}$$

From Theorem 1 and Lemma 4 due to $|\phi'|\geq 0$, $|\phi'(t)|$ is convex function, then by using Petrovic inequality, we get

$$\begin{aligned}&|\phi'(bt+(1-t)a)|+\left|\phi'\left(\frac{a+b}{2}-(bt+(1-t)a)\right)\right|\\&\leq\left|\phi'\left(bt+(1-t)a+\frac{a+b}{2}-(bt+(1-t)a)\right)\right|+|\phi'(0)|\\&=\left|\phi'\left(\frac{a+b}{2}\right)\right|+|\phi'(0)|,\end{aligned}\tag{2.4}$$

and

$$
\begin{aligned}
&\left|\phi'(bt+(1-t)a)\right|+\left|\phi'\left(bt+(1-t)a-\frac{a+b}{2}\right)\right| \\
&\leq \left|\phi'\left(bt+(1-t)a+bt+(1-t)a-\frac{a+b}{2}\right)\right|+\left|\phi'(0)\right| \\
&= \left|\phi'\left(2bt+2(1-t)a-\frac{a+b}{2}\right)\right|+\left|\phi'(0)\right| \qquad (2.5) \\
&= \left|\phi'\left(\left(2t-\frac{1}{2}\right)b+\left(\frac{3}{2}-2t\right)a\right)\right|+\left|\phi'(0)\right| \\
&\leq \left(2t-\frac{1}{2}\right)\left|\phi'(b)\right|+\left(\frac{3}{2}-2t\right)\left|\phi'(a)\right|+\left|\phi'(0)\right|.
\end{aligned}
$$

Considering (2.4) and (2.5) in (2.3), we get the following inequality

$$
\begin{aligned}
&\left|\phi\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b\left\{\phi(x)-\phi\left(\left|x-\frac{a+b}{2}\right|\right)\right\}dx\right| \\
&\leq (b-a)\int_0^{\frac{1}{2}} t\left\{\left|\phi'\left(\frac{a+b}{2}\right)\right|+\left|\phi'(0)\right|\right\}dt \\
&\quad +(b-a)\int_{\frac{1}{2}}^1(1-t)\left\{\left(2t-\frac{1}{2}\right)\left|\phi'(b)\right|+\left(\frac{3}{2}-2t\right)\left|\phi'(a)\right|+\left|\phi'(0)\right|\right\}dt+\phi(0) \\
&= \frac{(b-a)}{8}\left\{\left|\phi'\left(\frac{a+b}{2}\right)\right|+\left|\phi'(0)\right|\right\} \\
&\quad +\frac{(b-a)}{8}\left\{\frac{5}{6}\left|\phi'(b)\right|+\frac{1}{6}\left|\phi'(a)\right|+\left|\phi'(0)\right|\right\}+\phi(0) \\
&= \frac{(b-a)}{8}\left\{\left|\phi'\left(\frac{a+b}{2}\right)\right|+\frac{\left|\phi'(a)\right|+5\left|\phi'(b)\right|}{6}+2\left|\phi'(0)\right|\right\}+\phi(0) \\
&\leq \frac{(b-a)}{8}\left\{\frac{\left|\phi'(a)\right|+\left|\phi'(b)\right|}{2}+\frac{\left|\phi'(a)\right|+5\left|\phi'(b)\right|}{6}+2\left|\phi'(0)\right|\right\}+\phi(0) \\
&= \frac{(b-a)}{4}\left\{\frac{\left|\phi'(a)\right|+2\left|\phi'(b)\right|}{3}+\left|\phi'(0)\right|\right\}+\phi(0)
\end{aligned}
$$

and this is completed the proof.

Theorem 4. *Let* $\phi:[0,\infty)\to\mathbb{R}$ *be an integrable superquadratic function and* $0\leq a\leq b$ *and* $|\phi'|^r$ *be a superquadratic. Then, the following midpoint inequality holds:*

$$\left|\phi\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b\left\{\phi(x)-\phi\left(\left|x-\frac{a+b}{2}\right|\right)\right\}dx\right|$$
$$\leq\frac{(b-a)}{(p+1)^{\frac{1}{p}}2^{2(1+\frac{1}{r})}}\left[\left(|\phi'(b)|^r+3|\phi'(a)|^r\right)^{\frac{1}{r}}+\left(3|\phi'(b)|^r+|\phi'(a)|^r\right)^{\frac{1}{r}}\right]$$
$$+\frac{(b-a)}{2^{1+\frac{1}{r}}(p+1)^{\frac{1}{p}}}\left(|\phi'(0)|^r+\left|\phi'\left(\frac{b-a}{2}\right)\right|^r\right)^{\frac{1}{r}}+\phi(0)$$

where $r>1,\ \frac{1}{p}+\frac{1}{r}=1.$

Proof. From Lemma 4 , then by using Hölder Inequality, we get

$$\left|\phi\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b\left\{\phi(x)-\phi\left(\left|x-\frac{a+b}{2}\right|\right)\right\}dx\right|$$
$$\leq(b-a)\int_0^{\frac{1}{2}}t\left\{|\phi'(bt+(1-t)a)|+\left|\phi'\left(\frac{a+b}{2}-(bt+(1-t)a)\right)\right|\right\}dt$$
$$+(b-a)\int_{\frac{1}{2}}^1(1-t)\left\{|\phi'(bt+(1-t)a)|+\left|\phi'\left(bt+(1-t)a-\frac{a+b}{2}\right)\right|\right\}dt+\phi(0)$$
$$\leq(b-a)\left(\int_0^{\frac{1}{2}}t^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}|\phi'(bt+(1-t)a)|^r dt\right)^{\frac{1}{r}}$$
$$+(b-a)\left(\int_0^{\frac{1}{2}}t^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|\phi'\left(\frac{a+b}{2}-(bt+(1-t)a)\right)\right|^r dt\right)^{\frac{1}{r}}$$
$$+(b-a)\left(\int_{\frac{1}{2}}^1(1-t)^p\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1|\phi'(bt+(1-t)a)|^r dt\right)^{\frac{1}{r}}$$
$$+(b-a)\left(\int_{\frac{1}{2}}^1(1-t)^p\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|\phi'\left(bt+(1-t)a-\frac{a+b}{2}\right)\right|^r dt\right)^{\frac{1}{r}}+\phi(0).$$

Since $|\phi'|^r\geq 0,\ |\phi'|^r$ is convex function in Lemma 1 and using Hermite-Hadamard inequality, we have

$$\left|\phi\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b\left\{\phi(x)-\phi\left(\left|x-\frac{a+b}{2}\right|\right)\right\}dx\right|$$

$$\leq \frac{(b-a)}{(p+1)^{\frac{1}{p}}2^{\frac{p+1}{p}}}\left[\left(\int_0^{\frac{1}{2}}(t|\phi'(b)|^r+(1-t)|\phi'(a)|^r)dt\right)^{\frac{1}{r}}+\left(\frac{|\phi'(0)|^r+\left|\phi'\left(\frac{b-a}{2}\right)\right|^r}{4}\right)^{\frac{1}{r}}\right]$$

$$+\frac{(b-a)}{(p+1)^{\frac{1}{p}}2^{\frac{p+1}{p}}}\left[\left(\int_{\frac{1}{2}}^{1}(t|\phi'(b)|^r+(1-t)|\phi'(a)|^r)dt\right)^{\frac{1}{r}}+\left(\frac{|\phi'(0)|^r+\left|\phi'\left(\frac{b-a}{2}\right)\right|^r}{4}\right)^{\frac{1}{r}}\right]$$

$$+\phi(0)$$

$$=\frac{(b-a)}{(p+1)^{\frac{1}{p}}2^{\frac{p+1}{p}}}\left[\left(\frac{|\phi'(b)|^r+3|\phi'(a)|^r}{8}\right)^{\frac{1}{r}}+\left(\frac{|\phi'(0)|^r+\left|\phi'\left(\frac{b-a}{2}\right)\right|^r}{4}\right)^{\frac{1}{r}}\right.$$

$$\left.+\left(\frac{3|\phi'(b)|^r+|\phi'(a)|^r}{8}\right)^{\frac{1}{r}}+\left(\frac{|\phi'(0)|^r+\left|\phi'\left(\frac{b-a}{2}\right)\right|^r}{4}\right)^{\frac{1}{r}}\right]+\phi(0)$$

$$=\frac{(b-a)}{(p+1)^{\frac{1}{p}}2^{\frac{p+1}{p}}}\left[\left(\frac{|\phi'(b)|^r+3|\phi'(a)|^r}{8}\right)^{\frac{1}{r}}+\left(\frac{3|\phi'(b)|^r+|\phi'(a)|^r}{8}\right)^{\frac{1}{r}}\right.$$

$$\left.+2\left(\frac{|\phi'(0)|^r+\left|\phi'\left(\frac{b-a}{2}\right)\right|^r}{4}\right)^{\frac{1}{r}}\right]+\phi(0)$$

$$=\frac{(b-a)}{(p+1)^{\frac{1}{p}}2^{2(1+\frac{1}{r})}}\left[(|\phi'(b)|^r+3|\phi'(a)|^r)^{\frac{1}{r}}+(3|\phi'(b)|^r+|\phi'(a)|^r)^{\frac{1}{r}}\right]$$

$$+\frac{(b-a)}{2^{1+\frac{1}{r}}(p+1)^{\frac{1}{p}}}\left(|\phi'(0)|^r+\left|\phi'\left(\frac{b-a}{2}\right)\right|^r\right)^{\frac{1}{r}}+\phi(0)$$

and this completed the proof.

CONCLUSION

In this research, we have proved the Hermite-Hadamard inequalities for superquadratics functions. We establish the midpoint inequalities with using a important integral identity for differentiable superquadratic mappings. It is an interesting and new problem that the upcoming researchers may use the techniques of this research and prove fractional inequalities and similar inequalities or similiar our results can be obtained for superquadratics functions.

REFERENCES

1. S. Abramovich, G. Jameson, G. Sinnamon, Refining Jensen's inequality, *Bull. Math. Soc. Sci. Math. Roumanie* (N.S.) 47 (95) (2004) 3-14.
2. S. Abramovich, G. Jameson, and G. Sinnamon, Inequalities for averages of convex and superquadratic functions, *J. Inequal. Pure Appl. Math*, 5(4), 1-14, (2004).
3. S. Abramovich, On superquadraticity, *J. Math. Inequal.* 3(3), 329--339 (2009).
4. S. Abramovich, S. Ivelic, J. Pecaric, Improvement of Jensen--Steffensen's inequality for superquadratic functions, *Banach J. Math. Anal.* 4(1), 146-158 (2010).
5. R. Agarwal, *A propos d'une note de m. pierre humbert*, Comptes rendus de l'Academie des Sciences, vol. 236, no. 21, pp. 2031-2032, 1953.
6. S. Banic and S. Varosanec, Functional inequalities for superquadratic functions, *Int. J. Pure Appl. Math.* 43 (4) (2008), 537-549.
7. S. Banic, J. Pecaric, S. Varošanec, Superquadratic functions and refinements of some classical inequalities, *J. Korean Math. Soc.* 45 (2008) 513-525.
8. S. S. Dragomir and R.P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, *Appl. Math. Lett.*, 11(5) (1998), 91-95.
9. J. Hadamard, Etude sur les proprietes des fonctions entieres et en particulier d'une fonction considree par, *Riemann, J. Math. Pures. et Appl.* 58 (1893), 171-215.
10. U. S. Kirmaci, Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, *App. Math. and Comp.*, 147 (2004) 137-146.
11. M. Petrovic, Sur une equation fonctionnelle, *Publ. Math. Univ. Belgrade,* 1 (1932), 149-156.
12. A.W. Roberts and D.E. Varberg, *Convex Functions,* Academic Press, 300pp, New York, 1973.

Conformable Fractional Trapezoid Type Inequalities via s-Convex Functions

Mehmet Eyüp KİRİŞ[1], Gözde BAYRAK[1], Hüseyin BUDAK[2] and Hasan KARA[2]

[1]*Department of Mathematics, Faculty of Science and Arts, Afyon Kocatepe University, Afyonkarahisar-TURKEY*
[2]*Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce-TURKEY*

mkiris@gmail.com
gozdebayrak92@gmail.com
hsyn.budak@gmail.com
hasan64kara@gmail.com

ABSTRACT

In this article, some trapezoid-type inequalities are obtained for s-convex functions by means of conformable fractional integrals. These inequalities obtained are generalizations of inequalities for Riemann-Liouville fractional integrals and Riemann integrals.

1. INTRODUCTION AND PRELIMINARIES

The theory of convexity is an important study area of the literature. Research on convex functions is used in pure and applied mathematics. A formal definition for convex function may be stated as follows:

Definition 1. [6] *Let I be convex set on R. The function $f: I \to R$ is called convex on I, if it satisfies the following inequality:*

$$f(\eta x + (1-\eta)y) \leq \eta f(x) + (1-\eta) f(y) \tag{1.1}$$

for all $(x, y) \in I$ and $\eta \in 0,1]$. The mapping f is a concave on I if the inequality (1.1) holds in reversed direction for all $\eta \in 0,1]$ and $x, y \in I$.

Definition 2 *[4]Let $f: [0, \infty] \to R$ be a function and $0 < s \leq 1$. Then we have*

$$f(\mu x + \nu y) \leq \mu^s f(x) + \nu^s f(y) \tag{1.2}$$

for $\mu + \nu = 1$. The function f that provides this inequality is named the s -convex mapping in the second sense.

Remark 1 *If we take $s = 1$ in Definition 2, then Definition 2 reduce to Definition 1.*

Convex functions are widely used in integral inequalities. The Hermite-Hadamard inequality discovered by C. Hermite and J. Hadamard (see, e.g., [6], [17, p.137]) is one of the most well established inequalities in the theory of convex functions with a geometrical interpretation and many applications. This inequality states that if $f: I \to R$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \quad (1.3)$$

Both inequalities hold in the reversed direction if f is concave. Hermite-Hadamard inequality has been considered the most useful inequality in mathematical analysis. This inequality has been extended in a number of ways. For example, Dragomir and Agarwal first obtained trapezoid inequalities for convex functions in [5]. In [19], Sarikaya et al. generalized the inequalities (1.3) for fractional integrals and the authors also proved some corresponding trapezoid type inequalities.
Fractional integrals have been a focus of researchers in recent years. Before presenting some fractional integral definitions, let's give definitions of the gamma and the beta functions.

Definition 3. *The gamma function and the beta function are defined by*

$$\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}\,dt,$$

and

$$B(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\,dt,$$

respectively. Here, $0 < x, y < \infty$.

In [15], Kilbas et al. gave fractional integrals, also namely Riemann-Liouville integral operators as follows:

Definition 4. [15] *For $f \in L_1[a,b]$, the Riemann-Liouville integrals of order $\beta > 0$ are given by*

$$J_{a+}^{\beta}f(x) = \frac{1}{\Gamma(\beta)}\int_a^x (x-t)^{\beta-1}f(t)dt, \quad x > a \quad (1.4)$$

and

$$J_{b-}^{\beta}f(x) = \frac{1}{\Gamma(\beta)}\int_x^b (t-x)^{\beta-1}f(t)dt, \quad x < b, \quad (1.5)$$

respectively. The Riemann-Liouville integrals will be equal to their classical integrals for the condition $\beta = 1$.

The conformable fractional approach was developed, which depends on the fundamental definition of the derivative in [16]. In [2], the author proved that the conformable approach in

[16] cannot yield good results when compared to the Caputo definition for specific functions. This flaw in the conformable definition was avoided by some extensions of the conformable approach [8, 20]. Based on these approaches, Jarad et al. obtained the definitions of conformable fractional integrals in [11].

Definition 5. [11] *For $f \in L_1[a,b]$, the fractional conformable integral operator ${}^{\beta}J^{\alpha}_{a+}f$ and ${}^{\beta}J^{\alpha}_{b-}f$ of order $\beta > 0$ and $\alpha \in (0,1]$ are presented by*

$$ {}^{\beta}J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\beta)}\int_a^x \left(\frac{(x-a)^{\alpha}-(t-a)^{\alpha}}{\alpha}\right)^{\beta-1} \frac{f(t)}{(t-a)^{1-\alpha}}dt, \qquad t > a \tag{1.6} $$

and

$$ {}^{\beta}J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\beta)}\int_x^b \left(\frac{(b-x)^{\alpha}-(b-t)^{\alpha}}{\alpha}\right)^{\beta-1} \frac{f(t)}{(b-t)^{1-\alpha}}dt, \qquad t < b, \tag{1.7} $$

respectively.

If we consider $\alpha = 1$, then the fractional integral in (1.6) reduces to the Riemann-Liouville fractional integral in (1.4). Furthermore, the fractional integral in (1.7) coincides with the Riemann-Liouville fractional integral in (1.5) when $\alpha = 1$. For some recent results connected with fractional integral inequalities, see [1, 10] and the references cited therein.
Hyder et al. obtained the following equality that we will use in our principal outcomes.

Lemma 1. [3] *Consider that $f:[a,b] \to R$ is a differentiable function on (a,b) and $f' \in L[a,b]$. Then, for $\beta > 0$ and $\alpha \in (0,1]$, we obtain the following equality*

$$ \begin{aligned} &\frac{f(a)+f(b)}{2} - \frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(b-a)^{\alpha\beta}}\left[{}^{\beta}_{a}\Upsilon^{\alpha} f\left(\frac{a+b}{2}\right) {}^{\beta}\Upsilon^{\alpha}_{b} f\left(\frac{a+b}{2}\right)\right] \\ &= \frac{\alpha^{\beta}(b-a)}{4}\left[\int_0^1 \left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta} f'\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)dt \right. \\ &\left. - \int_0^1 \left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta} f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)dt\right]. \end{aligned} $$

Hyder et al. acquired some new trapezoid-type inequalities using conformable fractional integrals in [3]. Inspired by all the studies mentioned, we will esatblish some new trapezoid-type inequalities via s -convex functions based on conformable fractional integrals.

2. PRINCIPAL OUTCOMES

This section presents trapezoid-type inequalities type via differentiable s -convex functions.

Theorem 1. *Let $f:[a,b] \to R$ be a differentiable mapping on (a,b). If $|f'|$ is s -convex on $[a,b]$, then we get the following inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(b-a)^{\alpha\beta}}\left[{}_{a}^{\beta}\Upsilon^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{\beta}\Upsilon_{b}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right|$$
$$\leq\frac{b-a}{2^{s+2}}\left(\frac{1}{\alpha}B\left(\beta+1,\frac{s+1}{\alpha}\right)+\Psi(\alpha,\beta,s)\right)[|f'(a)|+|f'(b)|]$$

where

$$\Psi(\alpha,\beta,s):=\int_0^1 (1-(1-t)^{\alpha})^{\beta}(1+t)^{s}dt \tag{2.1}$$

and $B(\cdot,\cdot)$ refers to the beta function.

Proof. By taking modulus of Lemma 1, we acquire

$$\left|\frac{f(a)+f(b)}{2}-\frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(b-a)^{\alpha\beta}}\left[{}_{a}^{\beta}\Upsilon^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{\beta}\Upsilon_{b}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right| \tag{2.2}$$
$$\leq\frac{\alpha^{\beta}(b-a)}{4}\left[\int_0^1 \left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}\left|f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|dt\right.$$
$$\left.+\int_0^1 \left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt\right].$$

With help of the s -convexity of $|f'|$, we acquire

$$\left|\frac{f(a)+f(b)}{2}-\frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(b-a)^{\alpha\beta}}\left[{}_{a}^{\beta}\Upsilon^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{\beta}\Upsilon_{b}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right|$$
$$\leq\frac{\alpha^{\beta}(b-a)}{4\cdot 2^{s}}\left[\int_0^1 \left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}[(1-t)^{s}|f'(a)|+(1+t)^{s}|f'(b)|]dt\right.$$
$$\left.+\int_0^1 \left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}[(1-t)^{s}|f'(b)|+(1+t)^{s}|f'(a)|]\right].$$

By using the fact that

$$\int_0^1 [(1-(1-t)^{\alpha})^{\beta}](1-t)^{s}dt=\frac{1}{\alpha}B\left(\beta+1,\frac{s+1}{\alpha}\right), \tag{2.3}$$

we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(b-a)^{\alpha\beta}}\left[{}_{a}^{\beta}\Upsilon^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{\beta}\Upsilon_{b}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right|$$
$$\leq\frac{(b-a)}{4\cdot 2^{s}}\left(\left(\frac{1}{\alpha}B\left(\beta+1,\frac{s+1}{\alpha}\right)+\Psi(\alpha,\beta,s)\right)|f'(a)|+\left(\Psi(\alpha,\beta,s)+\right.\right.$$
$$\left.\left.\frac{1}{\alpha}B\left(\beta+1,\frac{s+1}{\alpha}\right)\right)|f'(b)|\right)$$
$$=\frac{(b-a)}{2^{s+2}}\left(\frac{1}{\alpha}B\left(\beta+1,\frac{s+1}{\alpha}\right)+\Psi(\alpha,\beta,s)\right)[|f'(a)|+|f'(b)|]$$

which completes the proof.

Corollary 1. *If we choose* $\alpha=1$ *in Theorem 1, then we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^{\beta}}\left[J_{a+}^{\beta}f\left(\frac{a+b}{2}\right)+J_{b-}^{\beta}f\left(\frac{a+b}{2}\right)\right]\right|$$
$$\leq\frac{b-a}{2^{s+2}}\big(B(\beta+1,s+1)+\Psi(1,\beta,s)\big)[|f'(a)|+|f'(b)|].$$

Remark 2. *If we take* $\alpha=1$ *and* $\beta=1$ *in Theorem 1, then we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq\frac{b-a}{2(s+1)(s+2)}\left(s+\frac{1}{2^s}\right)[|f'(a)|+|f'(b)|]$$

which is given by Kırmacı et al. in [14].

Remark 3. *In Theorem 1, if we choose* $s=1$*, then Theorem 1 reduce to [3, Theorem 5].*

Theorem 2. *Consider that* $f:[a,b]\to R$ *is a differentiable function on* (a,b)*. If* $|f'|^q$ *is* s *-convex on* $[a,b]$ *for* $q>1$*, then we establish the following inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(b-a)^{\alpha\beta}}\left[{}_{a}^{\beta}\Upsilon^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{\beta}\Upsilon_{b}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right|$$
$$\leq\frac{b-a}{4\cdot 2^{\frac{s}{q}}}\left(\frac{1}{\alpha}B\left(p\beta+1,\frac{1}{\alpha}\right)\right)^{\frac{1}{p}}$$
$$\times\left[\left(\frac{|f'(a)|^q+(2^{s+1}-1)|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{(2^{s+1}-1)|f'(a)|^q+|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}\right]$$

where $\frac{1}{p}=1-\frac{1}{q}$.

Proof. By using the Hölder's inequality, we obtain

$$\int_0^1\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}\left|f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|dt \tag{2.4}$$
$$\leq\left(\int_0^1\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|dt\right)^{\frac{1}{q}}.$$

From the s -convexity of $|f'|^q$, we get

$$\int_0^1\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}\left|f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|dt \tag{2.5}$$
$$\leq$$
$$\frac{1}{\alpha^{\beta}}\left(\int_0^1(1-(1-t)^{\alpha})^{p\beta}dt\right)^{\frac{1}{p}}\left(\int_0^1\left[\left(\frac{1-t}{2}\right)^s|f'(a)|^q+\left(\frac{1+t}{2}\right)^s|f'(b)|^q\right]dt\right)^{\frac{1}{q}}$$
$$=\frac{1}{2^{\frac{s}{q}}\cdot\alpha^{\beta}}\left(\frac{1}{\alpha}B\left(p\beta+1,\frac{1}{\alpha}\right)\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q+(2^{s+1}-1)|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}.$$

Similarly, we can write

$$\int_0^1 \left(\frac{1-(1-t)^\alpha}{\alpha}\right)^\beta \left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right| dt \le \frac{1}{2^{\frac{s}{q}}\cdot\alpha^\beta}\left(\frac{1}{\alpha}B\left(p\beta+1,\frac{1}{\alpha}\right)\right)^{\frac{1}{p}}\left(\frac{(2^{s+1}-1)|f'(a)|^q+|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}. \tag{2.6}$$

On substituting the inequalities (2.5) and (2.6) in (2.2), then the proof is accomplished.

Corollary 2. *If we choose $\alpha = 1$ in Theorem 2, then we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^\beta}\left[J_{a+}^\beta f\left(\frac{a+b}{2}\right)+J_{b-}^\beta f\left(\frac{a+b}{2}\right)\right]\right| \le \frac{b-a}{4\cdot 2^{\frac{s}{q}}}\left(\frac{p\beta}{p\beta+1}\right)^{\frac{1}{p}} \times\left[\left(\frac{|f'(a)|^q+(2^{s+1}-1)|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{(2^{s+1}-1)|f'(a)|^q+|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}\right].$$

Corollary 3. *If we take $\alpha = 1$ and $\beta = 1$ in Theorem 2, then we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{b-a}{4\cdot 2^{\frac{s}{q}}}\left(\frac{p}{p+1}\right)^{\frac{1}{p}} \times\left[\left(\frac{|f'(a)|^q+(2^{s+1}-1)|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{(2^{s+1}-1)|f'(a)|^q+|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}\right].$$

Remark 4. *If we allow $s = 1$ in Theorem 2, then Theorem 2 and [3, Theorem 6] are identical.*

Theorem 3. *Let us note that $f:[a,b]\to R$ is a differentiable function on (a,b). If $|f'|^q$ is s - convex function on $[a,b]$ for some $q \ge 1$, then we have the inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^\beta}{(b-a)^{\alpha\beta}}\left[{}^\beta_a\Upsilon^\alpha f\left(\frac{a+b}{2}\right)+{}^\beta\Upsilon^\alpha_b f\left(\frac{a+b}{2}\right)\right]\right| \le \frac{b-a}{2^{\frac{s}{q}+2}}\left(\frac{1}{\alpha}B\left(\beta+1,\frac{1}{\alpha}\right)\right)^{1-\frac{1}{q}} \times\left[\left(\frac{1}{\alpha}B\left(\beta+1,\frac{s+1}{\alpha}\right)|f'(a)|^q+\Psi(\alpha,\beta,s)|f'(b)|^q\right)^{\frac{1}{q}} +\left(\Psi(\alpha,\beta,s)|f'(a)|^q+\left(\frac{1}{\alpha}B\left(\beta+1,\frac{s+1}{\alpha}\right)\right)|f'(b)|^q\right)^{\frac{1}{q}}\right].$$

Here $\Psi(\alpha,\beta,s)$ is defined as in (2.1).

Proof. By using power mean inequality, we get

$$\int_0^1 \left(\frac{1-(1-t)^\alpha}{\alpha}\right)^\beta \left|f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right| dt$$
$$\leq \left(\int_0^1 \left|\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^\beta\right| dt\right)^{1-\frac{1}{q}}$$
$$\times \left(\int_0^1 \left|\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^\beta\right| \left|f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}.$$

Because of s -convexity of $|f'|^q$, we acquire

$$\int_0^1 \left(\frac{1-(1-t)^\alpha}{\alpha}\right)^\beta \left|f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right| dt \tag{2.7}$$
$$\leq \frac{1}{\alpha^\beta 2^{\frac{s}{q}}}\left(\frac{1}{\alpha}B\left(\beta+1,\frac{1}{\alpha}\right)\right)^{1-\frac{1}{q}}$$
$$\times \left(\int_0^1 (1-(1-t)^\alpha)^\beta[(1-t)^s|f'(a)|^q+(1+t)^s|f'(b)|^q]dt\right)^{\frac{1}{q}}$$
$$= \frac{1}{\alpha^\beta 2^{\frac{s}{q}}}\left(\frac{1}{\alpha}B\left(\beta+1,\frac{1}{\alpha}\right)\right)^{1-\frac{1}{q}}$$
$$\times \left(\left(\frac{1}{\alpha}B\left(\beta+1,\frac{s+1}{\alpha}\right)\right)|f'(a)|^q+\Psi(\alpha,\beta,s)|f'(b)|^q\right)^{\frac{1}{q}}.$$

Similarly, we get

$$\int_0^1 \left(\frac{1-(1-t)^\alpha}{\alpha}\right)^\beta \left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right| dt \tag{2.8}$$
$$\leq \frac{1}{\alpha^\beta 2^{\frac{s}{q}}}\left(\frac{1}{\alpha}B\left(\beta+1,\frac{1}{\alpha}\right)\right)^{1-\frac{1}{q}}$$
$$\times \left(\Psi(\alpha,\beta,s)|f'(a)|^q+\left(\frac{1}{\alpha}B\left(\beta+1,\frac{s+1}{\alpha}\right)\right)|f'(b)|^q\right)^{\frac{1}{q}}.$$

By considering (2.7) and (2.8) in (2.2), we obtain the required result.

Corollary 4. *If we choose* $\alpha=1$ *in Theorem 3, then we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^\beta}\left[J_{a+}^\beta f\left(\frac{a+b}{2}\right)+J_{b-}^\beta f\left(\frac{a+b}{2}\right)\right]\right|$$
$$\leq \frac{b-a}{2^{\frac{s}{q}+2}}\left(\frac{\beta}{\beta+1}\right)^{1-\frac{1}{q}}\left[\left(B(\beta+1,s+1)|f'(a)|^q+\Psi(1,\beta,s)|f'(b)|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\Psi(1,\beta,s)|f'(a)|^q+\left(B(\beta+1,s+1)\right)|f'(b)|^q\right)^{\frac{1}{q}}\right].$$

Corollary 5. *If we take* $\alpha = 1$ *and* $\beta = 1$ *in Theorem 3, then we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{b-a}{2^{\frac{s}{q}+2}}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\frac{1}{(s+1)(s+2)}|f'(a)|^q + \left(\frac{2^{s+2}-1}{s+1} - \frac{2^{s+1}-1}{s+2}\right)|f'(b)|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left(\frac{2^{s+2}-1}{s+1} - \frac{2^{s+1}-1}{s+2}\right)|f'(a)|^q + \frac{1}{(s+1)(s+2)}|f'(b)|^q\right)^{\frac{1}{q}}\right].$$

Remark 5. *If we set* $s = 1$ *in Theorem 3, then the Theorem 3 turns into* [3, Theorem 7].

3. CONCLUSION

In this research, we acquired some inequality of trapezoid type for s-convex functions by means of conformable fractional integrals. In the future studies, researchers can obtain some new inequalities with the aid of the different kinds of convex mappings or other types of fractional integral operators.

REFERENCES

[1] T. Abdeljawad, *On conformable fractional calculus*, J. Comput. Appl. Math. 279, 2015, 57-66.
[2] A. A. Abdelhakim, *The flaw in the conformable calculus: It is conformable because it is not fractional*, Fractional Calculus and Appied Analysis, 22, 2019, 242-254.
[3] A. A. Hyder, A. A. Almoneef, H. Budak, and M. A. Barakat, *On New Fractional Version of Generalized Hermite-Hadamard Inequalities*, Mathematics, 10(18), (2022) 3337.
[4] W. W. Breckner, *Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer Funktionen in topologischen linearen Räumen*, Publ. Inst. Math.(Beograd)(NS), 23(37), (1978), 13-20.
[5] S. S. Dragomir, R. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Applied mathematics letters, 11(5), (1998), 91-95.
[6] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000.
[7] H. R. Hwang, K. L. Tseng, and K. C. Hsu, New inequalities for fractional integrals and their applications, *Turkish Journal of Mathematics*, 40(3), (2016) 471–486.
[8] A. Hyder, A.H. Soliman, *A new generalized θ-conformable calculus and its applications in mathematical physics*, Physica Scripta, 96, 2020, 015208.
[9] M. Iqbal, S. Qaisar and M. Muddassar, *A short note on integral inequality of type Hermite–Hadamard through convexity*, J. Computational analysis and applications, 21(5), 2016, pp.946-953.
[10] F. Jarad, T. Abdeljawad, D. Baleanu, *On the generalized fractional derivatives and their Caputo modification*, J. Nonlinear Sci. Appl. 10(5), 2017, 2607-2619.

[11] F. Jarad, E. Uğurlu, T. Abdeljawad, and D. Baleanu, *On a new class of fractional operators*, Advances in Difference Equations, (2017), 2017, 247.
[12] H. Kara, H. Budak, and F. Hezenci, *New Extensions of the Parameterized Inequalities Based on Riemann–Liouville Fractional Integrals*. Mathematics, 10(18), 2022, 3374.
[13] U. S. Kirmaci, *Inequalities for differentiable mappings and applications to special means of real numbers to midpoint formula*, Appl. Math. Comput., vol. 147, no. 5, pp. 137–146, 2004.
[14] U. S. Kirmaci, M. K. Bakula, M. E. Özdemir, and J. Pečarić, *Hadamard-type inequalities for* s*-convex functions*, Applied Mathematics and Computation, 193(1), 2007, 26-35.
[15] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204, Elsevier Sci. B.V., Amsterdam, 2006.
[16] R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh, *A new definition of fractional derivative*, Journal of Computational and Applied Mathematics, 264, 2014, 65-70.
[17] J.E. Pecaric, F. Proschan and Y.L. Tong, *Convex Functions*, *Partial Orderings and Statistical Applications*, Academic Press, Boston, 1992.
[18] S. Özcan, İ. İscan, *Some new Hermite–Hadamard type inequalities for* s*-convex functions and their applications*. Journal of inequalities and applications, 2019(1), (2019). 1-11.
[19] M. Z. Sarikaya, E. Set, H. Yaldiz, N. Basak, *Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities*, Math Comput Model, 57 (9–10) (2013), 2403–2407.
[20] D. Zhao, M. Luo, *General conformable fractional derivative and its physical interpretation*, Calcolo, 54, 2017, 903-917.

Midpoint Type Inequalities Based on Conformable Fractional Integrals for s -Convex Mappings

Gözde BAYRAK[1], Mehmet Eyüp KİRİŞ[1], Mustafa Kemal YILDIZ[1] Hüseyin BUDAK[2] and Hasan KARA[2]

[1]*Department of Mathematics, Faculty of Science and Arts, Afyon Kocatepe University, Afyonkarahisar-TURKEY*
[2]*Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce-TURKEY*

gozdebayrak92@gmail.com
mkiris@gmail.com
myildiz@aku.edu.tr
hsyn.budak@gmail.com
hasan64kara@gmail.com

ABSTRACT

In the current research, some midpoint-type inequalities are acquired via s -convex mappings with the aid of conformable fractional integrals. Some studies in the literature have been generalized using the well-known Hölder and power-mean inequalities and s -convex mappings. Some results including Riemann-Liouville integrals and Riemann integrals established based on s -convex mappings by special choices of variables within functions are obtained

1. INTRODUCTION

Convex theory is a research area that has been utilized in many fields of optimization theory, energy systems, engineering applications, and physics and has guided many regions of the literature. Moreover, the convex theory is an available way to solve many problems from different branches of mathematics. Convexity theory is important in these branches of mathematics, especially in inequalities. Hermite-Hadamard, midpoint type, and trapezoid type inequalities are the most well-known of these inequalities.

These inequalities, described by C. Hermite and J. Hadamard, express that if $f: I \to R$ is a convex mapping on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

If f is concave, both of the inequalities hold in the opposite direction. See, please more references [7, 17]. The left side of the Hermite-Hadamard inequality, namely the midpoint inequality, has been the focus of many studies. Kirmac first, obtained midpoint inequalities for convex functions in [14]. Moreover in [18], Qaisar and Hussain presented several generalized midpoint type inequalities. Sarikaya et al. and Iqbal et al. proved some fractional trapezoid and midpoint type inequalities for convex functions in [19] and [11], respectively.In [4] and [5], researchers established some generalized midpoint type inequalities for Riemann-Liouville fractional integrals.

Fractional calculus is an effective tool to explain physical phenomena and also real-world problems. The concept of fractional order derivatives and integrals that will shed light on some unknown points about differential equations and solutions of some fractional order differential equations, which proved to be useless for their solution, is a novelty in applied sciences as well as mathematics. New derivatives and integrals contribute to the solution of differential equations that are expressed and solved in classical analysis, as well as fractional order derivatives and integrals. Moreover, it has increased its contribution to the literature with its applications in areas such as engineering, biostatistics, and mathematical biology. Fractional derivative and integral operators not only differed from each other in terms of singularity, locality, and kernels but also brought innovations to fractional analysis in terms of their usage areas and spaces.

The Conformable fractional approach was developed, which depends on the fundamental definition of the derivative in [16]. In [2], the author proved that the conformable approach in [16] cannot yield good results when compared to the Caputo definition for specific functions. This flaw in the conformable definition was avoided by some extensions of the conformable approach [10, 20]. Based on these approaches, Jarad obtained the definitions of conformable fractional integrals in [13]. Inspired by all these studies, fractional calculus attracts researchers every day.

Igbal et al. obtained some new midpoint-type inequalities in [11] with the help of Riemann-Liouville fractional integrals. Hyder et al obtained some new midpoint-type inequalities using conformable fractional integrals in [3]. Inspired by all the studies mentioned, we will obtain some new midpoint-type inequalities for s -convex functions with the aid of conformable fractional integrals.

2. PRELIMINARIES

In order to create our main results, in this section, we give the fundamental definitions and an identity.

Definition 1. *The gamma function and beta function are defined*

$$\Gamma(x):=\int_0^\infty t^{x-1}e^{-t}dt,$$

$$B(x,y):=\int_0^1 t^{x-1}(1-t)^{y-1}dt,$$

respectively. Here, $0<x,y<\infty$.

Definition 2. *[6]Let $f:[0,\infty]\to R$ be a function and $0<s\leq 1$. Then we have*

$$f(\lambda x+\gamma y)\leq\lambda^{s}f(x)+\gamma^{s}f(y) \tag{2.1}$$

for $\lambda+\gamma=1$. The function f that provides this inequality is called the s -convex function in the second sense.

Remark 1. *If we take $s=1$ in Definition 2, then Definition 2 reduce to deinition of classical convexity.*

In [15], Kilbas et al. presented fractional integrals, also namely Riemann-Liouville integral operators as follows:

Definition 3. *[15] For $f\in L_1[a,b]$, the Riemann-Liouville integrals of order $\beta>0$ are given by*

$$J_{a+}^{\beta}f(x)=\frac{1}{\Gamma(\beta)}\int_a^x (x-t)^{\beta-1}f(t)dt,\quad x>a \tag{2.2}$$

and

$$J_{b-}^{\beta}f(x)=\frac{1}{\Gamma(\beta)}\int_x^b (t-x)^{\beta-1}f(t)dt,\quad x<b, \tag{2.3}$$

respectively. The Riemann-Liouville integrals will be equal to their classical integrals for the condition $\beta=1$.

In paper [13], Jarad et al. gave the fractional conformable integral operators.

Definition 4. *[13] For $f\in L_1[a,b]$, the fractional conformable integral operator ${}^{\beta}J_{a+}^{\alpha}f(x)$ and $\beta J_{b-}^{\alpha}f(x)$ of order $\beta>0$ and $\alpha\in(0,1]$ are presented by*

$${}^{\beta}J_{a+}^{\alpha}f(x)=\frac{1}{\Gamma(\beta)}\int_a^x \left(\frac{(x-a)^{\alpha}-(t-a)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{f(t)}{(t-a)^{1-\alpha}}dt,\qquad t>a \tag{2.4}$$

and

$${}^{\beta}J_{b-}^{\alpha}f(x)=\frac{1}{\Gamma(\beta)}\int_x^b \left(\frac{(b-x)^{\alpha}-(b-t)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{f(t)}{(b-t)^{1-\alpha}}dt,\qquad t<b, \tag{2.5}$$

respectively.

If we consider $\alpha=1$, then the fractional integral in (2.4) reduces to the Riemann-Liouville fractional integral in (2.2). Furthermore, the fractional integral in (2.5) coincides with the Riemann-Liouville fractional integral in (2.3) when $\alpha=1$. For some recent results connected with fractional integral inequalities, see [1, 12] and the references cited therein.
Hyder et al. obtained the following identity that we will use in our main results.

Lemma 1. *[3] Let $f:[a,b]\to R$ be a differentiable mapping on (a,b) and $f'\in L^1[a,b]$. Then, for $\beta>0$ and $\alpha\in(0,1]$, the identity below is valid.*

$$\frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(b-a)^{\alpha\beta}}\left[{}_{a}^{\beta}\Upsilon^{\alpha}f\left(\frac{a+b}{2}\right)^{\beta}\Upsilon_{b}^{\alpha}f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right) \tag{2.6}$$

$$=\frac{\alpha^{\beta}(b-a)}{4}\left[\int_0^1\left[\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}\right]f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt\right.$$
$$\left.-\int_0^1\left[\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}\right]f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt\right].$$

3. MAIN RESULTS

This section provides numerous inequalities of the midpoint type for differentiable s -convex functions in the second sense.

Theorem 1. *Let $f:[a,b]\to R$ be a differentiable mapping on (a,b). If $|f'|$ is s -convex on $[a,b]$, then we get the inequality below.*

$$\left|\frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(b-a)^{\alpha\beta}}\left[{}_{a}^{\beta}\Upsilon^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{\beta}\Upsilon_{b}^{\alpha}f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{b-a}{2^{s+2}}\left(\frac{2^{s+1}}{s+1}-\frac{1}{\alpha}B\left(\beta+1,\frac{s+1}{\alpha}\right)-\Psi(\alpha,\beta,s)\right)[|f'(a)|+|f'(b)|] \tag{3.1}$$

where

$$\Psi(\alpha,\beta,s):=\int_0^1 (1-(1-t)^{\alpha})^{\beta}(1+t)^{s}dt \tag{3.2}$$

and $B(\cdot,\cdot)$ refers to the Euler Beta function.

Proof. By taking modulus of Lemma 1, we acquire

$$\left|\frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(b-a)^{\alpha\beta}}\left[{}_{a}^{\beta}\Upsilon^{\alpha}f\left(\frac{a+b}{2}\right)^{\beta}\Upsilon_{b}^{\alpha}f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right| \tag{3.3}$$
$$\leq\frac{\alpha^{\beta}(b-a)}{4}\left[\int_0^1\left|\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}\right|\left|f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|dt\right.$$
$$\left.+\int_0^1\left|\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}\right|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt\right].$$

With help of the s -convexity of $|f'|$, we have

$$\left|\frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(b-a)^{\alpha\beta}}\left[{}_{a}^{\beta}\Upsilon^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{\beta}\Upsilon_{b}^{\alpha}f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{\alpha^{\beta}(b-a)}{4\cdot 2^{s}}\left[\int_0^1\left|\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}\right|[(1-t)^{s}|f'(a)|+(1+t)^{s}|f'(b)|]dt\right.$$
$$\left.+\int_0^1\left|\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}\right|[(1-t)^{s}|f'(b)|+(1+t)^{s}|f'(a)|]\right]$$
$$=\frac{b-a}{4\cdot 2^{s}}\left(\int_0^1[1-(1-(1-t)^{\alpha})^{\beta}]((1-t)^{s}+(1+t)^{s})dt\right)[|f'(a)|+|f'(b)|].$$

By using the facts that

$$\int_0^1 \left[1-(1-(1-t)^\alpha)^\beta\right](1-t)^s dt = \frac{1}{s+1} - \frac{1}{\alpha} B\left(\beta+1, \frac{s+1}{\alpha}\right) \tag{3.4}$$

and

$$\begin{aligned} &\int_0^1 \left[1-(1-(1-t)^\alpha)^\beta\right](1+t)^s dt \\ &= \frac{2^{s+1}}{s+1} - \frac{1}{s+1} - \int_0^1 (1-(1-t)^\alpha)^\beta (1+t)^s dt \\ &= \frac{2^{s+1}-1}{s+1} - \Psi(\alpha,\beta,s), \end{aligned} \tag{3.5}$$

we have

$$\begin{aligned} &\left|\frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^\beta}{(b-a)^{\alpha\beta}}\left[{}_a^\beta\Upsilon^\alpha f\left(\frac{a+b}{2}\right) + {}^\beta\Upsilon_b^\alpha f\left(\frac{a+b}{2}\right)\right] - f\left(\frac{a+b}{2}\right)\right| \\ &\leq \frac{b-a}{2^{s+2}}\left(\frac{2^{s+1}}{s+1} - \frac{1}{\alpha}B\left(\beta+1,\frac{s+1}{\alpha}\right) - \Psi(\alpha,\beta,s)\right)[|f'(a)|+|f'(b)|] \end{aligned}$$

which completes the proof.

Corollary 1 *If we choose $\alpha = 1$ in Theorem 1, then we have*

$$\begin{aligned} &\left|\frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^\beta}\left[J_{a+}^\beta f\left(\frac{a+b}{2}\right) + J_{b-}^\beta f\left(\frac{a+b}{2}\right)\right] - f\left(\frac{a+b}{2}\right)\right| \\ &\leq \frac{b-a}{2^{s+2}}\left(\frac{2^{s+1}}{s+1} - B(\beta+1,s+1) - \Psi(1,\beta,s)\right)[|f'(a)|+|f'(b)|]. \end{aligned} \tag{3.6}$$

Remark 2. *If we take $s = 1$ in Corollary 1, then we have*

$$\begin{aligned} &\left|\frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^\beta}\left[J_{a+}^\beta f\left(\frac{a+b}{2}\right) + J_{b-}^\beta f\left(\frac{a+b}{2}\right)\right] - f\left(\frac{a+b}{2}\right)\right| \\ &\leq \frac{b-a}{4}\left(\frac{\beta}{\beta+1}\right)[|f'(a)|+|f'(b)|]. \end{aligned} \tag{3.7}$$

which is given by Ertuǧral et al. in [9, Corollary 4.7].

Remark 3. *Let us consider $\alpha = 1$ and $\beta = 1$ in Theorem 1, then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{(s+1)(s+2)}\left(1-\frac{1}{2^{s+1}}\right)[|f'(a)|+|f'(b)|].$$

which is given by Du et al. in [8, Theorem 2.1 (for m=k=1 and t=0)].

Remark 4. *In Theorem 1, if we choose $s = 1$, then Theorem 1 reduce to [3, Theorem 2].*

Theorem 2 *Consider that $f: [a,b] \to R$ is a differentiable function on (a,b). If $|f'|^q$ is s-convex on $[a,b]$ for $q > 1$, then we establish the following inequality*

$$\left|\frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(b-a)^{\alpha\beta}}\left[{}_{a}^{\beta}\Upsilon^{\alpha}f\left(\frac{a+b}{2}\right)^{\beta}\Upsilon_{b}^{\alpha}f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right| \tag{3.8}$$
$$\leq \frac{b-a}{4\cdot 2^{\frac{s}{q}}}\left(1-\frac{1}{\alpha}B\left(p\beta+1,\frac{1}{\alpha}\right)\right)^{\frac{1}{p}}$$
$$\times\left[\left(\frac{1}{s+1}|f'(a)|^{q}+\frac{2^{s+1}-1}{s+1}|f'(b)|^{q}\right)^{\frac{1}{q}}+\left(\frac{1}{s+1}|f'(a)|^{q}+\frac{2^{s+1}-1}{s+1}|f'(b)|^{q}\right)^{\frac{1}{q}}\right]$$

where $\frac{1}{p}=1-\frac{1}{q}$.

Proof. By using the equality (3.3) and Hölder's inequality, we obtain

$$\left|\frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(b-a)^{\alpha\beta}}\left[{}_{a}^{\beta}\Upsilon^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{\beta}\Upsilon_{b}^{\alpha}f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right| \tag{3.9}$$
$$\leq \frac{\alpha^{\beta}(b-a)}{4}\left[\left(\int_0^1\left|\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}\right|^{p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^{q}dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\left|\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}\right|^{p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^{q}dt\right)^{\frac{1}{q}}\right].$$

Because of s -convexity of $|f'|^{q}$, we get

$$\left(\int_0^1\left|\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}\right|^{p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^{q}dt\right)^{\frac{1}{q}} \tag{3.10}$$
$$\leq \frac{1}{\alpha^{\beta}}\left(\int_0^1\left(1-(1-(1-t)^{\alpha})^{\beta}\right)^{p}dt\right)^{\frac{1}{p}}$$
$$\times\left(\frac{1}{2^{s}}\int_0^1\left[\left(\frac{1-t}{2}\right)^{s}|f'(a)|^{q}+|f'(b)|^{q}\right]dt\right)^{\frac{1}{q}}$$
$$\leq \frac{1}{2^{\frac{s}{q}}\alpha^{\beta}}\left(\int_0^1\left(1-(1-(1-t)^{\alpha})^{\beta p}\right)dt\right)^{\frac{1}{p}}\left(\frac{1}{s+1}|f'(a)|^{q}+\frac{2^{s+1}-1}{s+1}|f'(b)|^{q}\right)^{\frac{1}{q}}$$
$$\leq \frac{1}{2^{\frac{s}{q}}\alpha^{\beta}}\left(1-\frac{1}{\alpha}B\left(p\beta+1,\frac{1}{\alpha}\right)\right)^{\frac{1}{p}}\left(\frac{1}{s+1}|f'(a)|^{q}+\frac{2^{s+1}-1}{s+1}|f'(b)|^{q}\right)^{\frac{1}{q}}.$$

Here, we used the fact that

$$(\varpi-\sigma)^{j}\leq\varpi^{j}-\sigma^{j}, \tag{3.11}$$

for any $\varpi>\sigma\geq 0$ and $j\geq 1$.
Similarly, we can write

$$\left(\int_0^1 \left|\frac{1}{\alpha^\beta}-\left(\frac{1-(1-t)^\alpha}{\alpha}\right)^\beta\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}} \tag{3.12}$$
$$\leq \frac{1}{2^{\frac{s}{q}}\alpha^\beta}\left(1-\frac{1}{\alpha}B\left(p\beta+1,\frac{1}{\alpha}\right)\right)^{\frac{1}{p}}\left(\frac{2^{s+1}-1}{s+1}|f'(a)|^q+\frac{1}{s+1}|f'(b)|^q\right)^{\frac{1}{q}}.$$

On substituting the inequalities (3.10) and (3.12) in (3.9), then the proof is accomplished.

Corollary 2. *If we choose $\alpha=1$ in Theorem 2, then we have*

$$\left|\frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^\beta}\left[J_{a+}^\beta f\left(\frac{a+b}{2}\right)+J_{b-}^\beta f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right| \tag{3.13}$$
$$\leq \frac{b-a}{4\cdot 2^{\frac{s}{q}}}\left(\frac{p\beta}{p\beta+1}\right)^{\frac{1}{p}}$$
$$\times\left[\left(\frac{1}{s+1}|f'(a)|^q+\frac{2^{s+1}-1}{s+1}|f'(b)|^q\right)^{\frac{1}{q}}+\left(\frac{1}{s+1}|f'(a)|^q+\frac{2^{s+1}-1}{s+1}|f'(b)|^q\right)^{\frac{1}{q}}\right].$$

Corollary 3. *If we choose $\alpha=1$ and $\beta=1$ in Theorem 2, then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\right| \tag{3.14}$$
$$\leq \frac{b-a}{4\cdot 2^{\frac{s}{q}}}\left(\frac{p}{p+1}\right)^{\frac{1}{p}}$$
$$\times\left[\left(\frac{1}{s+1}|f'(a)|^q+\frac{2^{s+1}-1}{s+1}|f'(b)|^q\right)^{\frac{1}{q}}+\left(\frac{1}{s+1}|f'(a)|^q+\frac{2^{s+1}-1}{s+1}|f'(b)|^q\right)^{\frac{1}{q}}\right].$$

Remark 5. *If we allow $s=1$ in Theorem 2, then Theorem 2 and [3, Theorem 3] are identical.*

Theorem 3 *Let us note that $f:[a,b]\to R$ is a differentiable function on (a,b). If $|f'|^q$ is s-convex on $[a,b]$ for some $q\geq 1$, then the inequality below is fulfilled.*

$$\left|\frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^\beta}{(b-a)^{\alpha\beta}}\left[{}_a^\beta\Upsilon^\alpha f\left(\frac{a+b}{2}\right)+{}^\beta\Upsilon_b^\alpha f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{b-a}{2^{\frac{s}{q}+2}}\left(1-\frac{1}{\alpha}B\left(\beta+1,\frac{1}{\alpha}\right)\right)^{1-\frac{1}{q}}$$
$$\times\left[\left(\left(\frac{1}{s+1}-\frac{1}{\alpha}B\left(\beta+1,\frac{s+1}{\alpha}\right)\right)|f'(a)|^q+\left(\frac{2^{s+1}-1}{s+1}-\Psi(\alpha,\beta,s)\right)|f'(b)|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left(\frac{2^{s+1}-1}{s+1}-\Psi(\alpha,\beta,s)\right)|f'(a)|^q+\left(\frac{1}{s+1}-\frac{1}{\alpha}B\left(\beta+1,\frac{s+1}{\alpha}\right)\right)|f'(b)|^q\right)^{\frac{1}{q}}\right].$$

Here $\Psi(\alpha,\beta,s)$ is defined as in (3.2).

Proof. With help of the equality (3.3) and by using power mean inequality,

$$\begin{aligned}
&\left|\frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(b-a)^{\alpha\beta}}\left[{}_{a}^{\beta}\Upsilon^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{\beta}\Upsilon_{b}^{\alpha}f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right| \\
&\leq \frac{\alpha^{\beta}(b-a)}{4}\left[\left(\int_0^1\left|\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}\right|dt\right)^{1-\frac{1}{q}}\right. \\
&\times\left(\int_0^1\left|\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}\right|\left|f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^{q}dt\right)^{\frac{1}{q}} \\
&+\left(\int_0^1\left|\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}\right|dt\right)^{1-\frac{1}{q}} \\
&\left.\times\left(\int_0^1\left|\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}\right|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^{q}dt\right)^{\frac{1}{q}}\right].
\end{aligned} \tag{3.15}$$

Taking into account the s -convexity of $|f'|^q$, then we acquire

$$\begin{aligned}
&\left(\int_0^1\left|\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}\right|dt\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1\left|\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}\right|\left|f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^{q}dt\right)^{\frac{1}{q}} \\
&\leq \frac{1}{\alpha^{\beta}2^{\frac{s}{q}}}\left(1-\frac{1}{\alpha}B\left(\beta+1,\frac{1}{\alpha}\right)\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1\left(1-(1-(1-t)^{\alpha})^{\beta}\right)[(1-t)^{s}|f'(a)|^{q}+(1+t)^{s}|f'(b)|^{q}]dt\right)^{\frac{1}{q}} \\
&=\frac{1}{\alpha^{\beta}2^{\frac{s}{q}}}\left(1-\frac{1}{\alpha}B\left(\beta+1,\frac{1}{\alpha}\right)\right)^{1-\frac{1}{q}} \\
&\times\left(\frac{1}{s+1}-\frac{1}{\alpha}B\left(\beta+1,\frac{s+1}{\alpha}\right)|f'(a)|^{q}+\left(\frac{2^{s+1}-1}{s+1}-\Psi(\alpha,\beta,s)\right)|f'(b)|^{q}\right),
\end{aligned} \tag{3.16}$$

and similarly, we have

$$\begin{aligned}
&\left(\int_0^1\left|\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}\right|dt\right)^{1-\frac{1}{q}} \\
&\left.\times\left(\int_0^1\left|\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}\right|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^{q}dt\right)^{\frac{1}{q}}\right] \\
&\leq \frac{1}{\alpha^{\beta}2^{\frac{s}{q}}}\left(1-\frac{1}{\alpha}B\left(\beta+1,\frac{1}{\alpha}\right)\right)^{1-\frac{1}{q}}
\end{aligned} \tag{3.17}$$

$$\times\left(\left(\frac{2^{s+1}-1}{s+1}-\Psi(\alpha,\beta,s)\right)|f'(a)|^q+\left(\frac{1}{s+1}-\frac{1}{\alpha}B\left(\beta+1,\frac{s+1}{\alpha}\right)\right)|f'(b)|^q\right).$$

By considering (3.16) and (3.17) in (3.15), we obtain the desired result.

Corollary 4 *If we choose $\alpha=1$ in Theorem 3, then we have*

$$\left|\frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^\beta}\left[J_{a+}^{\beta}f\left(\frac{a+b}{2}\right)+J_{b-}^{\beta}f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{b-a}{2^{\frac{s}{q}+2}}\left(\frac{\beta}{\beta+1}\right)^{1-\frac{1}{q}}$$
$$\times\left[\left(\left(\frac{1}{s+1}-B(\beta+1,s+1)\right)|f'(a)|^q\right.\right.$$
$$\left.+\left(\frac{2^{s+1}-1}{s+1}-\Psi(1,\beta,s)\right)|f'(b)|^q\right)^{\frac{1}{q}}$$
$$\left.+\left(\left(\frac{2^{s+1}-1}{s+1}-\Psi(1,\beta,s)\right)|f'(a)|^q+\left(\frac{1}{s+1}-B(\beta+1,s+1)\right)|f'(b)|^q\right)^{\frac{1}{q}}\right].$$

Corollary 5. *If we take $\alpha=1$ and $\beta=1$ in Theorem 3, then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{b-a}{2^{\frac{s}{q}+2}}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}$$
$$\times\left[\left(\left(\frac{1}{s+2}\right)|f'(a)|^q+\left(\frac{2^{s+2}-2}{s+1}-\frac{2^{s+2}-1}{s+2}\right)|f'(b)|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\left(\frac{2^{s+2}-2}{s+1}-\frac{2^{s+2}-1}{s+2}\right)|f'(a)|^q+\left(\frac{1}{s+2}\right)|f'(b)|^q\right)^{\frac{1}{q}}\right].$$

Remark 6. *If we set $s=1$ in Theorem 3, then Theorem 3 turns into [3, Theorem 4].*

CONCLUSION

In this study, some conformable fractional midpoint type inequalities in the case of s-convex functions are presented. Moreover, it is investigated several inequalities for Riemann-Liouville fractional integrals and Riemann integrals by choosing special cases of our main results. In future papers, improvement or generalization of our results can be investigated by using different kinds of convex function classes or other types fractional integral operators.

REFERENCES

[1] T. Abdeljawad, *On conformable fractional calculus*, J. Comput. Appl. Math. 279, 2015, 57-66.
[2] A. A. Abdelhakim, *The flaw in the conformable calculus: It is conformable because it is not fractional*, Fractional Calculus and Appied Analysis, 22, 2019, 242-254.
[3] A. A. Hyder, A. A. Almoneef, H. Budak, and M. A. Barakat, *On New Fractional Version of Generalized Hermite-Hadamard Inequalities*, Mathematics, 10(18), (2022) 3337.
[4] H. Budak and P. Agarwal, *New generalized midpoint type inequalities for fractional integral*, Miskolc Mathematical Notes, 20(2), 2019, 781-793.
[5] H. Budak and R. Kapucu, *New generalization of midpoint type inequalities for fractional integral*, An. S tiint. Univ Al. I. Cuza Ia s i. Mat. (N.S), 2021, 67(1).
[6] W. W. Breckner, *Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer Funktionen in topologischen linearen Räumen*, Publ. Inst. Math.(Beograd)(NS), 23(37), (1978), 13-20.
[7] S. S. Dragomir, R. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Applied mathematics letters, 11(5), (1998), 91-95.
[8] T. Du, Y. Li, and Z. Yang, *A generalization of Simpson's inequality via differentiable mapping using extended (s,m)-convex functions*, Applied mathematics and computation, 293, (2017) 358-369.
[9] F. Ertuğral, M. Z. Sar kaya, and H. Budak, *On Hermite-Hadamard type inequalities associated with the generalized fractional integrals*, Filomat, in press.
[10] A. Hyder, A.H. Soliman, *A new generalized θ-conformable calculus and its applications in mathematical physics*, Physica Scripta, 96, 2020, 015208.
[11] M. Iqbal, S. Qaisar and M. Muddassar, *A short note on integral inequality of type Hermite–Hadamard through convexity*, J. Computational analysis and applications, 21(5), 2016, pp.946-953.
[12] F. Jarad, T. Abdeljawad, D. Baleanu, *On the generalized fractional derivatives and their Caputo modification*, J. Nonlinear Sci. Appl. 10(5), 2017, 2607-2619.
[13] F. Jarad, E. U ğ urlu, T. Abdeljawad, and D. Baleanu, *On a new class of fractional operators*, Advances in Difference Equations, (2017), 2017, 247.
[14] U. S. Kirmaci, *Inequalities for differentiable mappings and applications to special means of real numbers to midpoint formula*, Appl. Math. Comput., vol. 147, 2004 no. 5, pp. 137–146.
[15] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204, Elsevier Sci. B.V., Amsterdam, 2006.
16] R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh, *A new definition of fractional derivative*, Journal of Computational and Applied Mathematics, 264, 2014, 65-70.

[17] S. Özcan, İ. İscan, *Some new Hermite–Hadamard type inequalities for s-convex functions and their applications*. Journal of inequalities and applications, 2019(1), (2019). 1-11.
[18] S. Qaisar and S. Hussain, *On Hermite–Hadamard type inequalities for functions whose first derivative absolute values are convex and concave*, Fasciculi Mathematici, 58(1), 2017, 155–166.
[19] M. Z. Sarikaya, E. Set, H. Yaldiz, N. Basak, *Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities*, Math Comput Model, 57 (9–10) (2013), 2403–2407.
[20] D. Zhao, M. Luo, *General conformable fractional derivative and its physical interpretation*, Calcolo, 54, 2017, 903-917.

Refinements of Hermite-Hadamard Inequalities for Conformable Fractional Integrals

Hüseyin BUDAK, Hasan KARA, Mehmet Zeki SARIKAYA

Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce-TURKEY

hsyn.budak@gmail.com
hasan64kara@gmail.com
sarikayamz@gmail.com

ABSTRACT

In the current paper, two new improvements for Hermite-Hadamard type inequalities are acquired with the help of the Conformable fractional integrals for convex functions. In achieving these improvements, two different functions are defined. More precisely, the convexity and increasing of the function are used. These improvements generalize some of the research in the literature.

1. INTRODUCTION

Fractional calculus has many applications in several different fields such as physics, chemistry, engineering, and mathematics. In terms of achieving more practical results in solving many problems, the application of arithmetic carried out in classical analysis in the fractional analysis is very significant. By using non-integer order dynamic models based on fractional computation, many practical dynamical systems are better characterized. Although integer orders in the classical analysis are a model that is not appropriate for nature, fractional computation in which arbitrary orders are studied helps us to obtain more practical approaches.

Fractional integral operators in a variety of scientific disciplines have been investigated widely. Using the derivative's fundamental limit formulation, a newly well-behaved straightforward fractional derivative known as the conformable derivative is improved in paper [11]. Some significant requirements that cannot be fulfilled by the Riemann-Liouville and Caputo definitions are fulfilled by the conformable derivative. However, in paper [2] the author proved that the conformable approach in [11] cannot yield good results when compared to the Caputo definition for specific functions. This flaw in the conformable definition was avoided by some extensions of the conformable approach [5, 17]. Based on these approaches, Jarad et al. obtained the definitions of conformable fractional integrals in [9]. Inspired by all these studies, fractional calculus attracts researchers every day.

The Hermite-Hadamard inequality is one of the most famous inequalities for convex functions in the literature. Some refinements of the Hermite–Hadamard inequality via convex mappings have been extensively obtained by a number researchers (see, [3, 12]). In [4], Dragomir presented an improvement for the first inequality of Hermite–Hadamard inequality. In [15], Yang and Hong obtained an improvement for the second inequality of the Hermite–Hadamard inequality. Sar kaya et al. acquired Hermite–Hadamard inequality involving Riemann-Liouville fractional integrals in [13]. By using an identity for both sides of this inequality obtained by Sar kaya et al., Xiang proved a new extension of this inequality in [14]. Set et al. offered a new Hermite–Hadamard inequality including conformable fractional integrals in [8]. In this investigation, we will present a new improvement for the first inequality of this expression obtained by Set et al., with the help of Jensen's inequality. We acquire a new extension for the second inequality of the Hermite–Hadamard type inequality based on conformable fractional integrals using the identity obtained by Xiang.

2. PRELIMINARIES

In order to create our main results, in this section, we present gamma function, beta function, definitions of Rieman-Liouville fractional integrals, and definitions of conformable fractional integrals.

Definition 1. *The gamma function and beta function are defined by*

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} dt,$$

and

$$B(x,y) := \int_0^1 t^{x-1} (1-t)^{y-1} dt,$$

respectivelly. Here, $0 < x, y < \infty$.

In [10], Kilbas et al. gave fractional integrals, also namely Riemann-Liouville fractional integral operators as follows:

Definition 2. [10] *For $f \in L_1[a,b]$, the Riemann-Liouville integrals of order $\beta > 0$ are given by*

$$J_{a+}^{\beta} f(x) = \frac{1}{\Gamma(\beta)} \int_a^x (x-t)^{\beta-1} f(t) dt, \quad x > a \tag{2.1}$$

and

$$J_{b-}^{\beta} f(x) = \frac{1}{\Gamma(\beta)} \int_x^b (t-x)^{\beta-1} f(t) dt, \quad x < b, \tag{2.2}$$

respectively. The Riemann-Liouville integrals will be equal to their classical integrals for the condition $\beta = 1$.

In [9], Jarad et al. gave the following fractional conformable integral operators.

Definition 3. [9] *For $f \in L_1[a,b]$, the fractional conformable integral operator ${}^{\beta}J_{a+}^{\alpha} f(x)$ and $\beta J_{b-}^{\alpha} f(x)$ of order$\beta \in C$, $Re(\beta) > 0$ and $\alpha \in (0,1]$ are presented by*

$${}^{\beta}J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\beta)} \int_a^x \left(\frac{(x-a)^{\alpha} - (t-a)^{\alpha}}{\alpha}\right)^{\beta-1} \frac{f(t)}{(t-a)^{1-\alpha}} dt, \qquad t > a \tag{2.3}$$

and

$$^{\beta}J_{b-}^{\alpha}f(x)=\frac{1}{\Gamma(\beta)}\int_x^b\left(\frac{(b-x)^{\alpha}-(b-t)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{f(t)}{(b-t)^{1-\alpha}}dt, \qquad t<b, \tag{2.4}$$

respectively.

Remark 1. *If we consider $\alpha=1$ in Definition 3, then Definition 3 reduces to Definition 2.*

For some recent results connected with fractional integral inequalities, see [1, 7] and the references cited therein.
Set et al. achieved a new Hermite–Hadamard inequality with the help of the conformable fractional integral operators.

Theorem 1. [8]*Note that f is a convex function on $[a,b]$. Then the following inequality is satisfied.*

$$f\left(\frac{a+b}{2}\right)\leq\frac{\Gamma(\beta+1)\alpha^{\beta}}{2(b-a)^{\alpha\beta}}\left[\,^{\beta}J_{b-}^{\alpha}f(a)+\,^{\beta}J_{a+}^{\alpha}f(b)\right]\leq\frac{f(a)+f(b)}{2}. \tag{2.5}$$

Here, $\beta>0,\ \alpha\in(0,1]$ and Γ is gamma function.

Theorem 2 (Weighted Jensen Inequality (WJI)). [6] *Let $f:[a,b]\to R$ be a convex function and let also $\varpi:[a,b]\to R^{+}$ and $\phi:[a,b]\to[a,b]$ be two integrable functions. Then we have*

$$f\left(\frac{1}{\int_a^b\varpi(x)dx}\int_a^b\varpi(x)\,\phi(x)dx\right)\leq\frac{1}{\int_a^b\varpi(x)dx}\int_a^b\varpi(x)\,f\big(\phi(x)\big)dx.$$

Xiang obtained the following equality that we will use in our main result.

Lemma 1. [16][14] *Consider $f:[a,b]\to R$ is a convex mapping and r be described by*

$$r(u)=\frac{1}{2}\left[f\left(\left(\frac{a+b}{2}\right)-\frac{u}{2}\right)+f\left(\left(\frac{a+b}{2}\right)+\frac{u}{2}\right)\right].$$

Then r is convex, increasing on $[0,b-a]$ and for all $u\in[0,b-a]$,

$$f\left(\frac{a+b}{2}\right)\leq r(u)\leq\frac{f(a)+f(b)}{2}.$$

3. MAIN RESULTS

In this section, we use conformable fractional integrals to obtain refinements of Hermite-Hadamard type inequalities.

Theorem 3. *Let $f:[a,b]\to R$ be a convex function and let $WB:[0,1]\to R$ be a function described by*

$$WB(t)=\frac{\alpha^{\beta}\beta}{2(b-a)^{\alpha\beta}}\int_a^b f\left(tx+(1-t)\frac{a+b}{2}\right)\Psi(x)dx.$$

Here, $\alpha\in[0,1]$, $\beta>0$, and

$$\Psi(x)=\left[\left(\frac{(b-a)^{\alpha}-(x-a)^{\alpha}}{\alpha}\right)^{\beta-1}(x-a)^{\alpha-1}+\left(\frac{(b-a)^{\alpha}-(b-x)^{\alpha}}{\alpha}\right)^{\beta-1}(b-x)^{\alpha-1}\right].$$

Then we have

1. WB is a convex mapping on $[0,1]$.
2. We have the following inequality:

$$f\left(\frac{a+b}{2}\right)\leq WB(t)\leq\frac{\alpha^{\beta}\Gamma(\beta+1)}{2(b-a)^{\alpha\beta}}\left[\ {}^{\beta}J_{b-}^{\alpha}f(a)+\ {}^{\beta}J_{a+}^{\alpha}f(b)\right] \qquad (3.1)$$

3. WB is monotonically increasing on $[0,1]$.

Proof. 1. Let $t_1,t_2\in[0,1]$ and $\lambda,\gamma\in[0,1]$ with $\lambda+\gamma=1$. Then we have

$$\begin{aligned}&WB(\lambda t_1+\gamma t_2)\\&=\frac{\alpha^{\beta}\beta}{2(b-a)^{\alpha\beta}}\int_a^b\left((\lambda t_1+\gamma t_2)x+\left(1-(\lambda t_1+\gamma t_2)\right)\frac{a+b}{2}\right)f\Psi(x)dx\\&=\frac{\alpha^{\beta}\beta}{2(b-a)^{\alpha\beta}}\int_a^b f\left(\lambda\left(t_1x+(1-t_1)\frac{a+b}{2}\right)+\gamma\left(t_2x+(1-t_2)\frac{a+b}{2}\right)\right)\Psi(x)dx.\end{aligned}$$

With the help of the convexity of f, we obtain

$$\begin{aligned}&WB(\lambda t_1+\gamma t_2)\\&\leq\frac{\alpha^{\beta}\beta}{2(b-a)^{\alpha\beta}}\int_a^b\left[\lambda f\left(t_1x+(1-t_1)\frac{a+b}{2}\right)+\gamma f\left(t_2x+(1-t_2)\frac{a+b}{2}\right)\right]\Psi(x)dx\\&=\lambda WB(t_1)+\gamma WB(t_2)\end{aligned}$$

from wich we have WB is convex on $[0,1]$.

2. Before employing the WJI, let's denote the following expressions

$$\begin{aligned}&\varpi(x)\\&=\frac{\alpha^{\beta}\beta}{2(b-a)^{\alpha\beta}}\left[\left(\frac{(b-a)^{\alpha}-(x-a)^{\alpha}}{\alpha}\right)^{\beta-1}(x-a)^{\alpha-1}+\left(\frac{(b-a)^{\alpha}-(b-x)^{\alpha}}{\alpha}\right)^{\beta-1}(b-x)^{\alpha-1}\right]\end{aligned}$$

$$= \frac{\alpha^{\beta}\beta}{2(b-a)^{\alpha\beta}} \Psi(x)$$

and

$$\phi(x) = tx + (1-t)\frac{a+b}{2}.$$

With the help of the change of variables $j = (b-a)^{\alpha} - (x-a)^{\alpha}$ and $k = (b-a)^{\alpha} - (b-x)^{\alpha}$, then we can write,

$$\int_a^b \Psi(x)\, dx = \left[\int_0^{(b-a)^{\alpha}} j^{\beta-1} dj + \int_0^{(b-a)^{\alpha}} k^{\beta-1} dk\right] \\ = \frac{2(b-a)^{\alpha\beta}}{\alpha^{\beta}\beta}. \tag{3.2}$$

Thus, we obtain

$$\int_a^b \varpi(x) dx = 1.$$

In addition, we have

$$\begin{aligned} &\int_a^b x\Psi(x)\, dx \\ &= \int_a^b (x-a+a)\left(\frac{(b-a)^{\alpha}-(x-a)^{\alpha}}{\alpha}\right)^{\beta-1}(x-a)^{\alpha-1}dx \\ &+ \int_a^b (x-b+b)\left(\frac{(b-a)^{\alpha}-(b-x)^{\alpha}}{\alpha}\right)^{\beta-1}(b-x)^{\alpha-1}dx \\ &= \int_a^b (x-a)^{\alpha}\left(\frac{(b-a)^{\alpha}-(x-a)^{\alpha}}{\alpha}\right)^{\beta-1} dx + a\frac{(b-a)^{\alpha\beta}}{\alpha^{\beta}\beta} \\ &+ b\frac{(b-a)^{\alpha\beta}}{\alpha^{\beta}\beta} - \int_a^b (b-x)^{\alpha}\left(\frac{(b-a)^{\alpha}-(b-x)^{\alpha}}{\alpha}\right)^{\beta-1} dx \\ &= \frac{(b-a)^{\alpha\beta}}{\alpha^{\beta}\beta}(a+b). \end{aligned} \tag{3.3}$$

Using the equalities (3.2) and (3.3), we can arrive at the following equality

$$\begin{aligned} &\int_a^b \phi(x)\, \varpi(x) dx \\ &= \frac{\alpha^{\beta}\beta}{2(b-a)^{\alpha\beta}} \int_a^b \left(tx + (1-t)\frac{a+b}{2}\right)\Psi(x)dx \\ &= \frac{\alpha^{\beta}t}{2(b-a)^{\alpha\beta}} \int_a^b x\Psi(x)dx + \frac{(a+b)(1-t)\alpha^{\beta}}{4(b-a)^{\alpha\beta}} \int_a^b \Psi(x)dx \\ &= \frac{a+b}{2}. \end{aligned}$$

By using WJI, we have

$$WB(t) = \frac{\alpha^{\beta}\beta}{2(b-a)^{\alpha\beta}} \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right)\Psi(x)dx$$

$$
\begin{aligned}
&= \int_a^b \varpi(x)\, f\big(\phi(x)\big)dx \\
&\geq \int_a^b \varpi(x)\, dx \cdot f\left(\frac{1}{\int_a^b \varpi(x)dx}\int_a^b \varpi(x)\,\phi(x)dx\right) \\
&= f\left(\frac{a+b}{2}\right)
\end{aligned}
$$

which completes the proof of first inequality of (3.1).

For the proof of the second inequality in (3.1), by using convexity of f, we get

$$
\begin{aligned}
WB(t) &= \frac{\alpha^\beta \beta}{2(b-a)^{\alpha\beta}}\int_a^b \left(tx+(1-t)\frac{a+b}{2}\right)\Psi(x)dx \\
&\leq \frac{\alpha^\beta \beta}{2(b-a)^{\alpha\beta}}\int_a^b tf(x)\Psi(x)dx+(1-t)f\left(\frac{a+b}{2}\right) \\
&= \frac{t\alpha^\beta \Gamma(\beta+1)}{2(b-a)^{\alpha\beta}}\left[\, {}^\beta J_{b-}^\alpha f(a)+\, {}^\beta J_{a+}^\alpha f(b)\right]+(1-t)f\left(\frac{a+b}{2}\right) \\
&:= \Upsilon(t).
\end{aligned}
$$

Taking the derivative of the function Υ,

$$\Upsilon'(t)=\frac{\alpha^\beta \Gamma(\beta+1)}{2(b-a)^{\alpha\beta}}\left[\, {}^\beta J_{b-}^\alpha f(a)+\, {}^\beta J_{a+}^\alpha f(b)\right]-f\left(\frac{a+b}{2}\right)$$

It is also seen from the inequality (2.5) that $\Upsilon'(t)\geq 0$. So the function Υ is increasing, such that

$$WB(t)\leq \Upsilon(t)\leq \Upsilon(1)=\frac{\alpha^\beta \Gamma(\beta+1)}{2(b-a)^{\alpha\beta}}\left[\, {}^\beta J_{b-}^\alpha f(a)+{}^\beta J_{a+}^\alpha f(b)\right].$$

Thus completes the proof of (3.1).

3. Since WB is convex on $[0,1]$, for $t_1,t_2\in[0,1]$ with $t_2>t_1$, we obtain

$$\frac{WB(t_2)-WB(t_1)}{t_2-t_1}\geq \frac{WB(t_1)-WB(0)}{t_1-0}=\frac{WB(t_1)-f\left(\frac{a+b}{2}\right)}{t_1}.$$

By first inequality in (3.1), we have $WB(t_1)\geq f\left(\frac{a+b}{2}\right)$, so we get

$$\frac{WB(t_2)-WB(t_1)}{t_2-t_1}\geq 0.$$

That is, $WB(t_2)\geq \Psi(t_1)$. This gives that WB is monotonically increasing on $[0,1]$.

So, the proof is accomplished.

Theorem 4. *Let f be described as in Theorem 3 and let $WK:[0,1]\to R$ is a function defined by*

$$WK(t) = \frac{\alpha^\beta}{4(b-a)^{\alpha\beta}} \int_a^b \left[f\left(\frac{1+t}{2}a + \frac{1-t}{2}x\right) \Xi_1(x) + f\left(\frac{1-t}{2}a + \frac{1+t}{2}x\right) \Xi_2(x) \right] dx$$

where

$$\Xi_1(x) = \left(\frac{(b-a)^\alpha - \left(b - \frac{a+x}{2}\right)^\alpha}{\alpha} \right)^{\beta-1} \left(b - \frac{a+x}{2}\right)^{\alpha-1} + \left[\frac{(b-a)^\alpha - \left(\frac{x-a}{2}\right)^\alpha}{\alpha} \right]^{\beta-1} \left(\frac{x-a}{2}\right)^{\alpha-1}$$

$$\Xi_2(x) = \left(\frac{(b-a)^\alpha - \left(\frac{b-x}{2}\right)^\alpha}{\alpha} \right)^{\beta-1} \left(\frac{b-x}{2}\right)^{\alpha-1} + \left[\frac{(b-a)^\alpha - \left(\frac{x+b}{2} - a\right)^\alpha}{\alpha} \right]^{\beta-1} \left(\frac{x+b}{2} - a\right)^{\alpha-1}.$$

Then we have

1. WK is a convex function on $[0,1]$.
2. WK is monotonically increasing on $[0,1]$.
3. We have the following inequality

$$\frac{\Gamma(\beta+1)\alpha^\beta}{2(b-a)^{\alpha\beta}} \left[{}^\beta J_{b-}^\alpha f(a) + {}^\beta J_{a+}^\alpha f(b) \right] \leq WK(t) \leq \frac{f(a)+f(b)}{2}.$$

Proof. 1. Let $t_1, t_2 \in [0,1]$ and $\lambda, \gamma \in [0,1]$ with $\lambda + \gamma = 1$. Then we have

$$\begin{aligned} &WK(\lambda t_1 + \gamma t_2) \\ &= \frac{\alpha^\beta}{4(b-a)^{\alpha\beta}} \int_a^b \left[f\left(\frac{1+\lambda t_1+\gamma t_2}{2}a + \frac{1-\lambda t_1-\gamma t_2}{2}x\right) \Xi_1(x) \right. \\ &\left. + f\left(\frac{1-\lambda t_1-\gamma t_2}{2}a + \frac{1+\lambda t_1+\gamma t_2}{2}x\right) \Xi_2(x) \right] dx \\ &= \frac{\alpha^\beta}{4(b-a)^{\alpha\beta}} \int_a^b f\left(\lambda\left(\left(\frac{1+t_1}{2}\right)a + \left(\frac{1-t_1}{2}\right)x\right) + \gamma\left(\left(\frac{1+t_2}{2}\right)a + \left(\frac{1-t_2}{2}\right)x\right) \right) [\Xi_1(x) \\ &+ f\left(\lambda\left(\left(\frac{1-t_1}{2}\right)a + \left(\frac{1+t_1}{2}\right)x\right) + \gamma\left(\left(\frac{1-t_2}{2}\right)a + \left(\frac{1+t_2}{2}\right)x\right) \right) \Xi_2(x) \Big] dx \end{aligned}$$

By using the convexity of f, then we have

$$\begin{aligned} &WK(\lambda t_1 + \gamma t_2) \\ &\leq \frac{\alpha^\beta}{4(b-a)^{\alpha\beta}} \int_a^b \left\{ \left[\lambda f\left(\left(\frac{1+t_1}{2}\right)a + \left(\frac{1-t_1}{2}\right)x\right) + \gamma f\left(\left(\frac{1+t_2}{2}\right)a + \left(\frac{1-t_2}{2}\right)x\right) \right] \Xi_1(x) \right. \\ &\left. + \left[\lambda f\left(\left(\frac{1-t_1}{2}\right)a + \left(\frac{1+t_1}{2}\right)x\right) + \gamma f\left(\left(\frac{1-t_2}{2}\right)a + \left(\frac{1+t_2}{2}\right)x\right) \right] \Xi_2(x) \right\} dx \\ &= \lambda WK(t_1) + \gamma WK(t_2). \end{aligned}$$

Hence, WK is convex on $[0,1]$.

2. By elemantary calculus, we have

$$
\begin{aligned}
&WK(t)\\
&=\frac{\alpha^{\beta}}{4(b-a)^{\alpha\beta}}\left[\int_a^b f\left(\frac{1+t}{2}a+\frac{1-t}{2}x\right)\Xi_1(x)dx+\int_a^b f\left(\frac{1-t}{2}a+\frac{1+t}{2}x\right)\Xi_2(x)dx\right]\\
&=\frac{\alpha^{\beta}}{4(b-a)^{\alpha\beta}}\left\{\int_0^{b-a} f\left(a+\frac{1-t}{2}x\right)\right.\\
&\times\left[\left(\frac{(b-a)^{\alpha}-\left(b-\frac{x+2a}{2}\right)^{\alpha}}{\alpha}\right)^{\beta-1}\left(b-\frac{x+2a}{2}\right)^{\alpha-1}\right.\\
&\left.+\left[\frac{(b-a)^{\alpha}-\left(\frac{x}{2}\right)^{\alpha}}{\alpha}\right]^{\beta-1}\left(\frac{x}{2}\right)^{\alpha-1}dx\right]\\
&+\int_0^{b-a} f\left(b-\frac{1-t}{2}x\right)\left[\left(\frac{(b-a)^{\alpha}-\left(\frac{x}{2}\right)^{\alpha}}{\alpha}\right)^{\beta-1}\left(\frac{x}{2}\right)^{\alpha-1}\right.\\
&\left.\left.+\left[\frac{(b-a)^{\alpha}-\left(\frac{2b-x}{2}-a\right)^{\alpha}}{\alpha}\right]^{\beta-1}\left(\frac{2b-x}{2}-a\right)^{\alpha-1}\right]dx\right\}\\
&=\frac{\alpha^{\beta}}{4(b-a)^{\alpha\beta}}\int_0^{b-a}\left[f\left(a+\frac{1-t}{2}x\right)+f\left(b-\frac{1-t}{2}x\right)\right]\Xi_3(x)dx,
\end{aligned}
$$

where

$$
\begin{aligned}
\Xi_3(x)&=\left(\frac{(b-a)^{\alpha}-\left(b-\frac{x+2a}{2}\right)^{\alpha}}{\alpha}\right)^{\beta-1}\left(b-\frac{x+2a}{2}\right)^{\alpha-1}\\
&+\left[\frac{(b-a)^{\alpha}-\left(\frac{x}{2}\right)^{\alpha}}{\alpha}\right]^{\beta-1}\left(\frac{x}{2}\right)^{\alpha-1}.
\end{aligned}
$$

It follows from Lemma 1 that $r(t)=\frac{1}{2}\left[f\left(\left(\frac{a+b}{2}\right)-\frac{t}{2}\right)+f\left(\left(\frac{a+b}{2}\right)+\frac{t}{2}\right)\right]$ and $\rho(t)=b-a-(1-t)x$ are increasing on $[0,b-a]$ and $[0,1]$, respectively. Hence $r(\rho(t))=f\left(a+\left(\frac{1-t}{2}\right)x\right)+f\left(b-\left(\frac{1-t}{2}\right)x\right)$ is increasing on $[0,1]$. Since $\Xi_3(x)$ is nonegative, it follows that WK is monotically increasing on $[0,1]$.

3. Since WK is monotically increasing on $[0,1]$, we get

$$
\begin{aligned}
&\frac{\Gamma(\beta+1)\alpha^{\beta}}{2(b-a)^{\alpha\beta}}\left[{}^{\beta}J_{b-}^{\alpha}f(a)+{}^{\beta}J_{a+}^{\alpha}f(b)\right]\\
&=WK(0)\le WK(t)\le WK(1)\\
&=\frac{f(a)+f(b)}{2}.
\end{aligned}
$$

So, the proof is completed.

4. CONCLUSION

In this current research, we present an extension of the Hermite-Hadamard inequality for the conformable fractional integrals. In the future, researchers can obtain new improvements of

Hermite-Hadamard inequality for different types of fractional integrals by utilizing the methods and techniques used in this paper. What's more, researchers can acquire new extensions of some inequalities obtained for different kinds of convexity, interval-valued functions, and quantum integrals. We hope that the ideas and techniques of this paper will inspire interested readers working in this field.

REFERENCES

[1] T. Abdeljawad, *On conformable fractional calculus*, J. Comput. Appl. Math. 279, 2015, 57-66.
[2] A. A. Abdelhakim, *The flaw in the conformable calculus: It is conformable because it is not fractional*, Fractional Calculus and Appied Analysis, 22, 2019, 242-254.
[3] M. U. Awan, S. Talib, Y. M. Chu, M. A. Noor, and K. I. Noor, *Some new refinements of Hermite–Hadamard-type inequalities involving-Riemann–Liouville fractional integrals and applications*, Mathematical Problems in Engineering, (2020).
[4] S. S. Dragomir, *Two mappings in connection to Hadamard's inequalities*, Journal of Mathematical Analysis and Applications, 167(1), (1992). 49-56.
[5] A. Hyder, A.H. Soliman, *A new generalized θ -conformable calculus and its applications in mathematical physics*, Physica Scripta, 96, 2020, 015208.
[6] B. G. Pachpatte, *Mathematical Inequalities*, Amsterdam, Holland: Elsevier Science, 2005, 67 ss. 11-45.
[7] F. Jarad, T. Abdeljawad, D. Baleanu, *On the generalized fractional derivatives and their Caputo modification*, J. Nonlinear Sci. Appl. 10(5), 2017, 2607-2619.
[8] E. Set, J. Choi, and A. Gözp nar, *Hermite-Hadamard Type Inequalities Involving Nonlocal Conformable Fractional Integrals*, Malaysian Journal of Mathematical Sciences 15(1), 2021, 33–43.
[9] F. Jarad, E. Uğurlu, T. Abdeljawad, and D. Baleanu, *On a new class of fractional operators*, Advances in Difference Equations, (2017), 2017, 247.
[10] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204, Elsevier Sci. B.V., Amsterdam, 2006.
[11] R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh, *A new definition of fractional derivative*, Journal of Computational and Applied Mathematics, 264, 2014, 65-70.
[12] F. Qi, Z. L. Wei, Q. Yang, *Generalizations and refinements of Hermite-Hadamard's inequality*, The Rocky Mountain Journal of Mathematics, 2005, 235-251, (2005).
[13] M. Z. Sarikaya, E. Set, H. Yaldiz, N. Basak, *Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities*, Math Comput Model, 57 (9–10) (2013), 2403–2407.
[14] R. Xiang, *Refinements of Hermite-Hadamard type inequalities for convex functions via fractional integrals*, Journal of applied mathematics & informatics, 33(1_2), (2015) 119-125.
[15] G. S. Yang and M. C. Hong, *A note on Hadamard's inequality*, Tamkang Journal of Mathematics, 28(1), 1997, 33-37.
[16] G. S. Yang and K. L. Tseng, *On certain integral inequalities related to Hermite–Hadamard inequalities*, Journal of Mathematical Analysis and Applications, 239(1), 1999, 180–187.
[17] D. Zhao, M. Luo, *General conformable fractional derivative and its physical interpretation*, Calcolo, 54, 2017, 903-917.

New Extensions of q_a-Hermite-Hadamard Inequality and q^b-Hermite-Hadamard Inequality

Hüseyin BUDAK

Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Türkiye

hsyn.budak@gmail.com

ABSTRACT

In this study, we first introduce two mappings depending quantum integrals. Then we show that these functions are convex and monotonically increasing. We also prove some refinements of the left-hand sides of the q_a-Hermite-Hadamard inequality and q^b-Hermite-Hadamard inequality.

1. INTRODUCTION

The Hermite-Hadamard inequality was proved by Hermite and Hadamard. It's one of the most recognized inequalities in the theory of convex functional analysis, which is stated as follows: Let $f:[a,b]\to R$ be a convex mapping on $[a,b]$. Then

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)\,dx\le\frac{f(a)+f(b)}{2}. \tag{1.1}$$

If f is concave, both inequalities hold in the reverse direction. Finding many studies in inequality theory, the quantum integral has gone through various searches by researchers to establish the quantum version of the famous Hermite-Hadamard inequality above. For the sake of brevity, let $q\in(0,1)$ and we use the following notation (see, [7]):

$$[n]_q=\frac{1-q^n}{1-q}=1+q+q^2+\ldots+q^{n-1}.$$

Definition 1. [12] The left quantum derivative or q_a -derivative of $f\ :\ [a,b]\to R$ at $x\in[a,b]$ is expressed as:

$${}_aD_qf(x)=\frac{f(x)-f(qx+(1-q)a)}{(1-q)(x-a)},x\neq a.$$

Definition 2. [3] The right quantum derivative or q^b -derivative of $f\ :\ [a,b]\to R$ at $x\in[a,b]$ is expressed as:

$${}^bD_qf(x)=\frac{f(qx+(1-q)b)-f(x)}{(1-q)(b-x)},x\neq b.$$

Definition 3. [12] The left quantum integral or q_a -integral of $f\ :\ [a,b]\to R$ at $x\in[a,b]$ is defined as:

$$\int_a^x f(t)\ {}_a d_q t = (1-q)(b-a)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a).$$

Definition 4. [3] The right quantum integral or q^b -integral of $f : [a,b] \to R$ at $x \in [a,b]$ is defined as:

$$\int_x^b f(t)\ {}^b d_q t = (1-q)(b-a)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)b).$$

In [2,3], Alp et al. and Bermudo et al. derive two different versions of q -Hermite-Hadamard inequalities and some estimates with the help of the q -derivatives and integrals. The q -Hermite-Hadamard inequalities are stated as:

Theorem 1. [2,3] For a convex mapping $f: [a,b] \to R$, the following inequalities hold:

$$f\left(\frac{qa+b}{[2]_q}\right) \le \frac{1}{b-a}\int_a^b f(x)\ {}_a d_q x \le \frac{qf(a)+f(b)}{[2]_q}, \tag{1.2}$$

$$f\left(\frac{a+qb}{[2]_q}\right) \le \frac{1}{b-a}\int_a^b f(x)\ {}^b d_q x \le \frac{f(a)+qf(b)}{[2]_q}. \tag{1.3}$$

Remark 1. It is very easy to observe that by adding (1.2) and (1.3), we have following q-Hermite-Hadamard inequality (see, [3]):

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{2(b-a)}\left[\int_a^b f(x)\ {}_a d_q x + \int_a^b f(x)\ {}^b d_q x\right] \le \frac{f(a)+f(b)}{2}. \tag{1.4}$$

Hereabout, Ali et al. [1] and Sitthiwirattham et al. [11] utilized calculates to present the following two different and new versions of Hermite-Hadamard type inequalities:

Theorem 2. [8,9] For a convex mapping $f : [a,b] \to R$, the following inequalities hold:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} f(x)\ {}^{\frac{a+b}{2}} d_q x + \int_{\frac{a+b}{2}}^b f(x)\ {}_{\frac{a+b}{2}} d_q x\right] \le \frac{f(a)+f(b)}{2}, \tag{1.5}$$

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} f(x)\ {}_a d_q x + \int_{\frac{a+b}{2}}^b f(x)\ {}^b d_q x\right] \le \frac{f(a)+f(b)}{2}. \tag{1.6}$$

Remark 2. If we allow limit as $q \to 1^-$ in (1.2)-(1.6), then the inequalities (1.2)-(1.6) reduce to classical Hermite-Hadamard inequality (1.1).

A lot of research has been done on q-integral inequalities with the help of different convexities. For instance, in [4], some new midpoint and trapezoidal type inequalities for q-integrals and q -differentiable convex functions were established. For more recent inequalities in q-calculus, one can consult [5, 8-10, 12, 13].

The main goal of the paper is to define two functions including quantum integrals. Then we prove the convexity and monotony of these functions. With the help of the newly presented functions, we also acquire some improvement of the left-hand sides of the inequalities of q_a-Hermite-Hadamard type and inequality of q^b-Hermite-Hadamard type.

2. MAIN RESULTS

In this section, we will define two functions including quantum integrals. We will prove how these functions are functions that improve the Hermite-Hadamard inequalities (1.5) and (1.6).

Theorem 3. Let $f : [a,b] \to R$ be a convex function and let $\Psi : [0,1] \to R$ be a function defined by

$$\Psi(t) = \frac{1}{b-a}\int_a^{\frac{a+b}{2}} f\left(tx + (1-t)\frac{(1+2q)a+b}{2[2]_q}\right)\ {}_a d_q x + \frac{1}{b-a}\int_{\frac{a+b}{2}}^{b} f\left(tx + (1-t)\frac{a+(1+2q)b}{2[2]_q}\right)\ {}^b d_q x.$$

Then we have;

1) Ψ is convex on $[0,1]$.
2) We have the following inequality:

$$f\left(\frac{a+b}{2}\right) \le \Psi(t) \le \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} f(x)\ {}_a d_q x + \int_{\frac{a+b}{2}}^{b} f(x)\ {}^b d_q x\right]. \tag{2.1}$$

3) Ψ is monotonically increasing on $[0,1]$.

Proof 1). Let $t,s \in [0,1]$ and $\alpha,\beta \in [0,1]$ with $\alpha+\beta=1$. Then we have

$$\begin{aligned}\Psi(\alpha t+\beta s) &= \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} f\left((\alpha t+\beta s)x + (1-(\alpha t+\beta s))\frac{(1+2q)a+b}{2[2]_q}\right)\ {}_a d_q x\right.\\ &\quad \left.+\int_{\frac{a+b}{2}}^{b} f\left((\alpha t+\beta s)x + (1-(\alpha t+\beta s))\frac{a+(1+2q)b}{2[2]_q}\right)\ {}^b d_q x\right]\\ &= \frac{1}{b-a}\int_a^{\frac{a+b}{2}} f\left(\alpha\left(tx+(1-t)\frac{(1+2q)a+b}{2[2]_q}\right)+\beta\left(sx+(1-s)\frac{(1+2q)a+b}{2[2]_q}\right)\right)\ {}_a d_q x\\ &\quad +\frac{1}{b-a}\int_{\frac{a+b}{2}}^{b} f\left(\alpha\left(tx+(1-t)\frac{a+(1+2q)b}{2[2]_q}\right)+\beta\left(sx+(1-s)\frac{a+(1+2q)b}{2[2]_q}\right)\right)\ {}^b d_q x.\end{aligned}$$

By using the convexity of f, we derive

$$\begin{aligned}\Psi(\alpha t+\beta s) &\le \frac{1}{b-a}\int_a^{\frac{a+b}{2}} \left[\alpha f\left(tx+(1-t)\frac{(1+2q)a+b}{2[2]_q}\right)+\beta f\left(sx+(1-s)\frac{(1+2q)a+b}{2[2]_q}\right)\right]\ {}_a d_q x\\ &\quad +\frac{1}{b-a}\int_{\frac{a+b}{2}}^{b} \left[\alpha f\left(tx+(1-t)\frac{a+(1+2q)b}{2[2]_q}\right)+\beta f\left(sx+(1-s)\frac{a+(1+2q)b}{2[2]_q}\right)\right]\ {}^b d_q x\\ &= \frac{\alpha}{b-a}\int_a^{\frac{a+b}{2}} f\left(tx+(1-t)\frac{(1+2q)a+b}{2[2]_q}\right)\ {}_a d_q x + \frac{\alpha}{b-a}\int_{\frac{a+b}{2}}^{b} f\left(tx+(1-t)\frac{a+(1+2q)b}{2[2]_q}\right)\ {}^b d_q x\end{aligned}$$

$$+\frac{\beta}{b-a}\int_a^{\frac{a+b}{2}} f\left(sx+(1-s)\frac{(1+2q)a+b}{2[2]_q}\right)\ {}_ad_qx+\frac{\beta}{b-a}\int_{\frac{a+b}{2}}^{b} f\left(sx+(1-s)\frac{a+(1+2q)b}{2[2]_q}\right)\ {}^bd_qx$$
$$=\alpha\Psi(t)+\beta\Psi(s).$$

Hence, Ψ is convex on $[0,1]$.

2). By Definition of q_a-integral and q^b-integral, we have

$$\begin{aligned}\Psi(t)&=\frac{1}{b-a}\int_a^{\frac{a+b}{2}} f\left(tx+(1-t)\frac{(1+2q)a+b}{2[2]_q}\right)\ {}_ad_qx\\&\quad+\frac{1}{b-a}\int_{\frac{a+b}{2}}^{b} f\left(tx+(1-t)\frac{a+(1+2q)b}{2[2]_q}\right)\ {}^bd_qx\\&=\frac{1-q}{2}\sum_{n=0}^{\infty}q^n f\left(t\left(q^n\frac{a+b}{2}+(1-q^n)a\right)+(1-t)\frac{(1+2q)a+b}{2[2]_q}\right)\\&\quad+\frac{1-q}{2}\sum_{n=0}^{\infty}q^n f\left(t\left(q^n\frac{a+b}{2}+(1-q^n)b\right)+(1-t)\frac{a+(1+2q)b}{2[2]_q}\right).\end{aligned}$$

Since $\sum_{n=0}^{\infty}(1-q)q^n=1$, by using Jensen inequality, we establish

$$\begin{aligned}\Psi(t)&\ge\frac{1}{2}f\left(\sum_{n=0}^{\infty}(1-q)q^n\left(t\left(\frac{q^nb}{2}+\frac{(2-q^n)a}{2}\right)+(1-t)\frac{(1+2q)a+b}{2[2]_q}\right)\right)\\&\quad+\frac{1}{2}f\left(\sum_{n=0}^{\infty}(1-q)q^n\left(t\left(\frac{q^na}{2}+\frac{(2-q^n)b}{2}\right)+(1-t)\frac{a+(1+2q)b}{2[2]_q}\right)\right)\\&=\frac{1}{2}f\left(t\left(\frac{b}{2[2]_q}+\frac{(1+2q)a}{2[2]_q}\right)+(1-t)\frac{(1+2q)a+b}{2[2]_q}\right)+\frac{1}{2}f\left(t\left(\frac{a}{2[2]_q}+\frac{(1+2q)b}{2[2]_q}\right)+(1-t)\frac{a+(1+2q)b}{2[2]_q}\right)\\&=\frac{1}{2}\left[f\left(\frac{(1+2q)a+b}{2[2]_q}\right)+f\left(\frac{a+(1+2q)b}{2[2]_q}\right)\right]\\&\ge f\left(\frac{a+b}{2}\right)\end{aligned}$$

which proves first inequality in (2.1). For the proof of second inequality, by using convexity of f, we get

$$\begin{aligned}\Psi(t)&=\frac{1}{b-a}\int_a^{\frac{a+b}{2}} f\left(tx+(1-t)\frac{(1+2q)a+b}{2[2]_q}\right)\ {}_ad_qx\\&\quad+\frac{1}{b-a}\int_{\frac{a+b}{2}}^{b} f\left(tx+(1-t)\frac{a+(1+2q)b}{2[2]_q}\right)\ {}^bd_qx\\&\le\frac{1}{b-a}\int_a^{\frac{a+b}{2}}\left[tf(x)+(1-t)f\left(\frac{(1+2q)a+b}{2[2]_q}\right)\right]\ {}_ad_qx\\&\quad+\frac{1}{b-a}\int_{\frac{a+b}{2}}^{b}\left[tf(x)+(1-t)f\left(\frac{a+(1+2q)b}{2[2]_q}\right)\right]\ {}^bd_qx\\&=\frac{t}{b-a}\left(\int_a^{\frac{a+b}{2}} f(x)\ {}_ad_qx+\int_{\frac{a+b}{2}}^{b} f(x)\ {}^bd_qx\right)+\frac{1-t}{2}\left[f\left(\frac{(1+2q)a+b}{2[2]_q}\right)+f\left(\frac{a+(1+2q)b}{2[2]_q}\right)\right]\\&:=w(t).\end{aligned}$$

By applying the inequalities (1.2) and (1.3) for the intervals $\left[a,\frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2},b\right]$, respectively, we have the inequalities

$$f\left(\frac{(1+2q)a+b}{2[2]_q}\right)\leq\frac{2}{b-a}\int_a^{\frac{a+b}{2}}f(x)\ {}_ad_qx \tag{2.2}$$

and

$$f\left(\frac{a+(1+2q)b}{2[2]_q}\right)\leq\frac{2}{b-a}\int_{\frac{a+b}{2}}^{b}f(x)\ {}^bd_qx. \tag{2.3}$$

It is clear from the inequalities (2.2) and (2.3) that w is monotonically increasing on $[0,1]$. Therefore we have

$$\Psi(t)\leq w(t)\leq w(1)=\frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}}\ f(x)\ {}_ad_qx+\int_{\frac{a+b}{2}}^{b}f(x)\ {}^bd_qx\right].$$

This finishes the proof of (2.1).

3). Since Ψ is convex on $[0,1]$, for $t_1,t_2\in[0,1]$ with $t_2>t_1$, we obtain

$$\frac{\Psi(t_2)-\Psi(t_1)}{t_2-t_1}\geq\frac{\Psi(t_1)-\Psi(0)}{t_1-0}=\frac{1}{t_1}\left(\Psi(t_1)-\frac{1}{2}\left[f\left(\frac{(1+2q)a+b}{2[2]_q}\right)+f\left(\frac{a+(1+2q)b}{2[2]_q}\right)\right]\right)\geq\frac{1}{t_1}\left(\Psi(t_1)-f\left(\frac{a+b}{2}\right)\right). \tag{2.4}$$

The last inequality in (2.4) is clear from the convexity of f. By first inequality in (2.1), we have $\Psi(t_1)\geq f\left(\frac{a+b}{2}\right)$, so we get

$$\frac{\Psi(t_2)-\Psi(t_1)}{t_2-t_1}\geq 0.$$

That is $\Psi(t_2)\geq\Psi(t_1)$. This gives that Ψ monotonically increasing on $[0,1]$.

Theorem 4. Let $f:[a,b]\to R$ be a convex function and let $\Upsilon:[0,1]\to R$ be a mapping defined by

$$\Upsilon(t)=\frac{1}{b-a}\int_a^{\frac{a+b}{2}}f\left(tx+(1-t)\frac{(2+q)a+qb}{2[2]_q}\right)\ {}^{\frac{a+b}{2}}d_qx$$
$$+\frac{1}{b-a}\int_{\frac{a+b}{2}}^{b}f\left(tx+(1-t)\frac{qa+(2+q)b}{2[2]_q}\right)\ {}_{\frac{a+b}{2}}d_qx$$

Then we have;
1) Υ is convex on $[0,1]$.
2) We have the following inequality:

$$f\left(\frac{a+qb}{[2]_q}\right)\leq\Upsilon(t)\leq\frac{1}{b-a}\int_a^b f(x)\ {}^bd_qx. \tag{2.5}$$

3) Υ is monotonically increasing on $[0,1]$.

Proof 1). Let us consider $t, s \in [0,1]$ and $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$. Then we get

$$\begin{aligned}
\Upsilon(\alpha t+\beta s) &= \frac{1}{b-a}\int_a^{\frac{a+b}{2}} f\left((\alpha t+\beta s)x+\left(1-(\alpha t+\beta s)\right)\frac{(2+q)a+qb}{2[2]_q}\right)\ {}_{\frac{a+b}{2}}d_qx \\
&\quad + \frac{1}{b-a}\int_{\frac{a+b}{2}}^{b} f\left((\alpha t+\beta s)x+\left(1-(\alpha t+\beta s)\right)\frac{qa+(2+q)b}{2[2]_q}\right)\ {}_{\frac{a+b}{2}}d_qx \\
&= \frac{1}{b-a}\int_a^{\frac{a+b}{2}} f\left(\alpha\left(tx+(1-t)\frac{(2+q)a+qb}{2[2]_q}\right)+\beta\left(sx+(1-s)\frac{(2+q)a+qb}{2[2]_q}\right)\right)\ {}_{\frac{a+b}{2}}d_qx \\
&\quad + \frac{1}{b-a}\int_{\frac{a+b}{2}}^{b} f\left(\alpha\left(tx+(1-t)\frac{qa+(2+q)b}{2[2]_q}\right)+\beta\left(sx+(1-s)\frac{qa+(2+q)b}{2[2]_q}\right)\right)\ {}_{\frac{a+b}{2}}d_qx.
\end{aligned}$$

Since f is convex on $[a, b]$, we can write

$$\begin{aligned}
\Upsilon(\alpha t+\beta s) &\le \frac{1}{b-a}\int_a^{\frac{a+b}{2}} \left[\alpha f\left(tx+(1-t)\frac{(2+q)a+qb}{2[2]_q}\right)+\beta f\left(sx+(1-s)\frac{(2+q)a+qb}{2[2]_q}\right)\right]\ {}_{\frac{a+b}{2}}d_qx \\
&\quad + \frac{1}{b-a}\int_{\frac{a+b}{2}}^{b} \left[\alpha f\left(tx+(1-t)\frac{qa+(2+q)b}{2[2]_q}\right)+\beta f\left(sx+(1-s)\frac{qa+(2+q)b}{2[2]_q}\right)\right]\ {}_{\frac{a+b}{2}}d_qx \\
&= \alpha\Upsilon(t)+\beta\Upsilon(s).
\end{aligned}$$

Therefore, Υ is convex on $[0,1]$.

2). Since $\sum_{n=0}^{\infty}(1-q)q^n = 1$, by using Jensen inequality, we establish

$$\begin{aligned}
\Upsilon(t) &= \frac{1}{b-a}\int_a^{\frac{a+b}{2}} f\left(tx+(1-t)\frac{(2+q)a+qb}{2[2]_q}\right)\ {}_{\frac{a+b}{2}}d_qx \\
&\quad + \frac{1}{b-a}\int_{\frac{a+b}{2}}^{b} f\left(tx+(1-t)\frac{qa+(2+q)b}{2[2]_q}\right)\ {}_{\frac{a+b}{2}}d_qx \\
&= \frac{(1-q)}{2}\sum_{n=0}^{\infty} q^n f\left(t\left(q^n a+(1-q^n)\frac{a+b}{2}\right)+(1-t)\frac{(2+q)a+qb}{2[2]_q}\right) \\
&\quad + \frac{(1-q)}{2}\sum_{n=0}^{\infty} q^n f\left(t\left(q^n b+(1-q^n)\frac{a+b}{2}\right)+(1-t)\frac{qa+(2+q)b}{2[2]_q}\right) \\
&\ge \frac{1}{2} f\left(\sum_{n=0}^{\infty}(1-q)q^n\left(t\left(\frac{1+q^n}{2}a+\frac{1-q^n}{2}b\right)+(1-t)\frac{(2+q)a+qb}{2[2]_q}\right)\right) \\
&\quad + \frac{1}{2} f\left(\sum_{n=0}^{\infty}(1-q)q^n\left(t\left(\frac{1+q^n}{2}b+\frac{1-q^n}{2}a\right)+(1-t)\frac{qa+(2+q)b}{2[2]_q}\right)\right) \\
&= \frac{1}{2} f\left(t\left(\frac{(2+q)a+qb}{2[2]_q}\right)+(1-t)\frac{(2+q)a+qb}{2[2]_q}\right)+\frac{1}{2} f\left(t\left(\frac{(2+q)b+qa}{2[2]_q}\right)+(1-t)\frac{qa+(2+q)b}{2[2]_q}\right) \\
&= \frac{1}{2}\left[f\left(\frac{(2+q)a+qb}{2[2]_q}\right)+f\left(\frac{(2+q)b+qa}{2[2]_q}\right)\right] \\
&\ge f\left(\frac{a+q}{2}\right).
\end{aligned}$$

This proves first inequality in (2.5). Since f is convex on $[a, b]$, we have

$$
\begin{aligned}
\Upsilon(t) &= \frac{1}{b-a}\int_a^{\frac{a+b}{2}} f\left(tx+(1-t)\frac{(2+q)a+qb}{2[2]_q}\right)\ {}_a d_q x \\
&+\frac{1}{b-a}\int_{\frac{a+b}{2}}^{b} f\left(tx+(1-t)\frac{qa+(2+q)b}{2[2]_q}\right)\ {}_{\frac{a+b}{2}}d_q x \\
&\leq \frac{1}{b-a}\int_a^{\frac{a+b}{2}}\left[tf(x)+(1-t)f\left(\frac{(2+q)a+qb}{2[2]_q}\right)\right]\ {}_a d_q x \\
&+\frac{1}{b-a}\int_{\frac{a+b}{2}}^{b}\left[tf(x)+(1-t)f\left(\frac{qa+(2+q)b}{2[2]_q}\right)\right]\ {}_{\frac{a+b}{2}}d_q x \\
&= \frac{t}{b-a}\left[\int_a^{\frac{a+b}{2}} f(x)\ {}_a d_q x+\int_{\frac{a+b}{2}}^{b} f(x)\ {}_{\frac{a+b}{2}}d_q x\right]+\frac{1-t}{2}\left[f\left(\frac{(2+q)a+qb}{2[2]_q}\right)+f\left(\frac{qa+(2+q)b}{2[2]_q}\right)\right] \\
&:= g(t).
\end{aligned}
$$

It is clear from the inequality (1.2) and (1.3) that g is monotonically increasing on $[0,1]$. Therefore we have

$$
\Upsilon(t) \leq g(t) \leq g(1) = \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} f(x)\ {}_a d_q x+\int_{\frac{a+b}{2}}^{b} f(x)\ {}_{\frac{a+b}{2}}d_q x\right].
$$

This finishes the proof of (2.5).

3). Since Υ is convex on $[0,1]$, for $t_1, t_2 \in [0,1]$ with $t_2 > t_1$, we obtain

$$
\frac{\Upsilon(t_2)-\Upsilon(t_1)}{t_2-t_1} \geq \frac{\Upsilon(t_1)-\Upsilon(0)}{t_1-0} = \frac{\Upsilon(t_1)-f\left(\frac{a+qb}{[2]_q}\right)}{t_1}.
$$

By the first inequality in (2.5), we have $\Upsilon(t_1) \geq f\left(\frac{a+qb}{[2]_q}\right)$, so we get

$$
\frac{\Upsilon(t_2)-\Upsilon(t_1)}{t_2-t_1} \geq 0.
$$

That is, $\Upsilon(t_2) \geq \Upsilon(t_1)$. This gives that Υ is monotonically increasing on $[0,1]$.

REFERENCES

[1] M. A. Ali, H. Budak, M. Fečkon and S. Khan, A new version of q-Hermite-Hadamard's midpoint and trapezoid type inequalities for convex functions. *Mathematica Slovaca, 2022, in press.*

[2] N. Alp, M. Z. Sarikaya, M. Kunt and İ. İşcan, q-Hermite Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions. *J. King Saud Univ. Sci.*, **30** (2018), 193--203.

[3] S. Bermudo, P. Kórus and J. N. Valdés, On q-Hermite--Hadamard inequalities for general convex functions. *Acta Math. Hung.*, **162** (2020), 364-374.

[4] H. Budak, Some trapezoid and midpoint type inequalities for newly defined quantum integrals. *Proyecciones*, **40** (2021), 199-215.

[5] H. Gauchman, Integral inequalities in q -calculus. *Comput. Math. Appl.,* **47** (2004), 281-300.

[6] V. Kac and P. Cheung *Quantum calculus*. Springer (2001).

[7] M. A. Noor, K. I. Noor and M. U. Awan, Some quantum estimates for Hermite-Hadamard inequalities. *Appl. Math. Comput.*, **251** (2015), 675--679.

[8] M. A. Noor, K. I. Noor and M. U. Awan, Some quantum integral inequalities via preinvex functions. *Appl. Math. Comput.,* **269** (2015), 242--251.

[9] W. Sudsutad, S. K. Ntouyas and J. Tariboon, Quantum integral inequalities for convex functions. *J. Math. Inequal.,* **9** (2015), 781--793.

[10] T. Sitthiwirattham, M. A. Ali, A. Ali and H. Budak, A new q -Hermite-Hadamard's inequality and estimates for midpoint type inequalities for convex functions. *Miskolc Mathematical Notes*, 2022, in press.

[11] J. Tariboon and S. K. Ntouyas, Quantum calculus on finite intervals and applications to impulsive difference equations. *Adv. Difference Equ.,* **282** (2013), 1-19.

[12] H. Zhuang, W. Liu and J. Park, Some quantum estimates of Hermite-Hadamard inequalities for quasi-convex functions, *Mathematics,* **7** (2019), 152.

Analysis of A TB Mathematical Model via Fractional Operator

Mustafa Ali DOKUYUCU[1], Tuba OĞUR[2]

[1]*Department of Mathematics, Faculty of Science and Arts,*
Ağrı İbrahim Çeçen University, Ağrı-TÜRKİYE
[2]*Graduate School of Education,*
Ağrı İbrahim Çeçen University, Ağrı-TURKEY

mustafaalidokuyucu@gmail.com
tubaogur34@gmail.com

ABSTRACT

The public health is at risk due to the bacillus Mycobacterium tuberculosis infection that causes tuberculosis (TB). TB-infected individuals typically spread the disease through the air when they speak, sneeze, cough, or spit. This illness affects the human body's lungs the most frequently, but it can also spread to other organs like the brain, spine, kidneys, and central nervous system. In this context, the analysis of the TB mathematical model will be made through fractional derivative operators. The solution of the existence of the model to be extended to the fractional derivative operator will be examined. Then, the uniqueness of the solution of the mathematical model will be investigated.

1. INTRODUCTION

Diseases that spread quickly across large geographic areas are known as epidemics. For thousands of years, these diseases have been a significant issue for humanity. To examine how these diseases arise, mathematical models have been developed. This can give information on the disease's pace of spread, its contagiousness, the total number of cases, and the estimated total number of fatalities brought on by the sickness.

A contagious illness called tuberculosis can infect your lungs or other tissues. The organs most frequently affected by it are the lungs, but it can also harm your spine, brain, or kidneys. The Latin root of the word "tuberculosis" means "nodule" or "anything that stands out." A second name for tuberculosis is TB. Not everyone who contracts TB becomes ill, but if you do, you need to get treated. If you have the bacterium but no symptoms, you have latent tuberculosis, also known as dormant tuberculosis (also called latent TB). Although it may appear that TB has disappeared, it is actually dormant (sleeping) inside your body. You have active tuberculosis or tuberculosis if you are infected, experience symptoms, and are spreadable.

In this study, we will analyze the tuberculosis mathematical model using the fractional derivative operator. First, we will extend the mathematical model to the fractional derivative operator. We will then examine the existence and uniqueness of the solution of the model.

2. PRELIMINARIES

Descriptions and theorems regarding the non-singular fractional Caputo-Fabrizio operator are presented in this part. Please see [1,2,3] articles for more detailed information.

Definition 2.1. The well-known fractional order Caputo derivative is defined as follows [1], let $f \in H^1(a,b)$

$$ {}_a^C D_t^\rho f(t) = \frac{1}{\Gamma(n-\rho)} \int_a^t \frac{f^{(n)}(r)}{(t-r)^{\rho+1-n}}, \tag{2.1}$$

where $n-1<\rho<n \in N$.

Definition 2.2. Let $f \in H^1(a,b), 0<\rho<1$. The new Caputo fractional derivative is defined as follows [2],

$$ {}_a^{CF} D_t^\rho f(t) = \frac{\rho M(\rho)}{1-\rho} \int_a^t \frac{df(x)}{dx} \, exp\left[\rho \frac{x-t}{1-\rho}\right] dx, \tag{2.2}$$

Here $M(\rho)$ is a normalization constant. Also $M(0)$ and $M(1)$ are equal to 1. Further it can be written below, if the f does not belong to $H^1(a,b)$.

$$ {}_a^{CF} D_t^\rho f(t) = \frac{\rho M(\rho)}{1-\rho} \int_a^t \big(f(t)-f(x)\big) exp\left[\rho \frac{x-t}{1-\rho}\right] dx. \tag{2.3}$$

Definition 2.3. Let $f \in H^1(a,b)$, $0<\rho<1$. The Caputo-Fabrizio fractional derivative of order f is as follows [3],

$$ {}^{CF}D_\star^\rho f(t) = \frac{1}{1-\rho} \int_a^t f'(x) exp\left[\rho \frac{x-t}{1-\rho}\right] dx. \tag{2.4}$$

Definition 2.4. Let $0<\rho<1$. The fractional integral order ρ of a function f is defined by [3],

$$ I^\rho f(t) = \frac{2(1-\rho)}{(2-\rho)M(\rho)} u(t) + \frac{2\rho}{(2-\rho)M(\rho)} \int_a^t u(s)ds \tag{2.5}$$

3. MAIN RESULTS

The original normalized TB virus model can be described by the following equations [4]:

$$\begin{aligned}
\frac{dS}{dt} &= \wedge - \frac{\beta SI}{N} - \mu S \\
\frac{dL}{dt} &= \frac{\beta SI}{N} - (\mu + \varepsilon) L (1-\eta)\delta T \\
\frac{dI}{dt} &= \varepsilon L + \eta \delta T - (\mu + \gamma + \sigma_1) I \\
\frac{dT}{dt} &= \gamma I - (\mu + \delta + \sigma_2 + \xi) T \\
\frac{dR}{dt} &= \xi T - \mu R
\end{aligned} \tag{3.1}$$

We present the proposed fractional model to describe the dynamics of TB infection. To develop the model, total human population is divided into five epidemiological sub-compartments denoted by susceptible S(t), Exposed L(t), TB active I(t), under treatment T(t), and recovered individuals after treatment R(t).

Equations could be written as in the form of Caputo Fabrizio fractional derivative:

$$\begin{aligned}
{}^{CF}_{t}D_0^a S(t) &= \wedge - \frac{\beta SI}{N} - \mu S \\
{}^{CF}_{t}D_0^a L(t) &= \frac{\beta SI}{N} - (\mu + \varepsilon) L (1-\eta)\delta T \\
{}^{CF}_{t}D_0^a I(t) &= \varepsilon L + \eta \delta T - (\mu + \gamma + \sigma_1) I \\
{}^{CF}_{t}D_0^a T(t) &= \gamma I - (\mu + \delta + \sigma_2 + \xi) T \\
{}^{CF}_{t}D_0^a R(t) &= \xi T - \mu R
\end{aligned} \tag{3.2}$$

In this section we will give the existence and uniqueness of the solutions [5]. Now applying the fractional integral in equation (3.2), and let initial values are $S_0(0) = S(0)$, $L_0(0) = L(0)$, $I_0(0) = I(0)$, $T_0(0) = T(0)$, $R_0(0) = R(0)$. now we obtain the following,

$$\begin{aligned}
S(t) - S(0) &= \frac{2(1-a)}{2M(a) - aM(a)} \left(\wedge - \frac{\beta S(t) I(t)}{N} - \mu S(t)\right) + \\
&\quad \frac{2a}{2M(a) - aM(a)} \int_0^t \left(\wedge - \frac{\beta S(y) I(y)}{N} - \mu S(y)\right) dy \\
L(t) - L(0) &= \frac{2(1-a)}{2M(a) - aM(a)} \left(\frac{\beta S(t) I(t)}{N} - (\mu + \varepsilon) L(t)(1-\eta)\delta T(t)\right) + \\
&\quad \frac{2a}{2M(a) - aM(a)} \int_0^t \frac{\beta S(y) I(y)}{N} - (\mu + \varepsilon) L(y)(1-\eta)\delta T(y) dy \\
I(t) - I(0) &= \frac{2(1-a)}{2M(a) - aM(a)} \left(\varepsilon L(t) + \eta \delta T(t) - (\mu + \gamma + \sigma_1) I(t)\right) + \\
&\quad \frac{2a}{2M(a) - aM(a)} \int_0^t \left(\varepsilon L(y) + \eta \delta T(y) - (\mu + \gamma + \sigma_1) I(y)\right) dy \\
T(t) - T(0) &= \frac{2(1-a)}{2M(a) - aM(a)} \left(\gamma I(t) - (\mu + \delta + \sigma_2 + \xi) T(t)\right) + \\
&\quad \frac{2a}{2M(a) - aM(a)} \int_0^t \left(\gamma I(y) - (\mu + \delta + \sigma_2 + \xi) T(y)\right) dy \\
R(t) - R(0) &= \frac{2(1-a)}{2M(a) - aM(a)} \left(\xi T(t) - \mu R(t)\right) + \\
&\quad \frac{2a}{2M(a) - aM(a)} \int_0^t \left(\xi T(y) - \mu R(y)\right) dy
\end{aligned} \tag{3.3}$$

For simplicity, we define function A_i and some constants $\gamma_{i,}$ $i = 1, \dots, 5$

$$
\begin{aligned}
&A_1(t,S) = \wedge - \frac{\beta S(t)I(t)}{N} - \mu S(t) \\
&A_2(t,L) = \frac{\beta S(t)I(t)}{N} - (\mu+\varepsilon)L(t)(1-\eta)\delta T(t) \\
&A_3(t,I) = \varepsilon L(t) + \eta\delta T(t) - (\mu+\gamma+\sigma_1)I(t) \\
&A_4(t,T) = \gamma I(t) - (\mu+\delta+\sigma_2+\xi)T(t) \\
&A_5(t,R) = \xi T(t) - \mu R(t)
\end{aligned} \tag{3.4}
$$

$$
\begin{aligned}
&\gamma_1 = \frac{\beta\varepsilon_3}{N} + \mu \\
&\gamma_2 = (\mu+\varepsilon)(1-\eta)\delta\varepsilon_3 \\
&\gamma_3 = \mu+\gamma+\sigma_1 \\
&\gamma_4 = \mu+\delta+\sigma_2+\xi \\
&\gamma_5 = \mu
\end{aligned} \tag{3.5}
$$

For proving our results for the following continuous functions $F: S(t),\ S_1(t),\ L(t),\ L_1(t),\ I(t),\ I_1(t),\ T(t),\ T_1(t),\ R(t)\ and\ R_1(t)$, such that, $\|S(t)\| \le \varepsilon_1$, $\|L(t)\| \le \varepsilon_2$, $\|I(t)\| \le \varepsilon_{3,}$ $\|T(t)\| \le \varepsilon_4\ and\ \|R(t)\| \le \varepsilon_5$.

Theorem 3.1. The kernels $A_{i,i=1,..,5}$ are satisfying the Lipschitz condition if the contractions provided $\gamma_i < 1, i = 1,..,5$.

Proof. First, we prove that $A_1(t,S)$ satisfies Lipschitz condition. For S(t) and $S_1(t)$ using equation (3.4) we have,

$$
\begin{aligned}
\|A_1(t,S) - A_1(t,S_1)\| &= \left\|(\wedge - \frac{\beta SI}{N} - \mu S) - (\wedge - \frac{\beta S_1 I}{N} - \mu S_1)\right\| \\
&= \left\|(S-S_1)(\frac{\beta I}{N} + \mu)\right\| \\
&\le (\frac{\beta\varepsilon_3}{N} + \mu)\|S-S_1\| \\
&= \gamma_1\|S-S_1\|
\end{aligned} \tag{3.6}
$$

Second, we show that $A_2\,(t,L)$ satisfies Lipschitz condition. For L(t) and L_1(t) using equation (3.4), we obtain

$$
\begin{aligned}
\|A_2(t,L) - A_2(t,L_1)\| &= \left\|(\frac{\beta SI}{N} - (\mu+\varepsilon)L(1-\eta)\delta T) - (\frac{\beta SI}{N} - (\mu+\varepsilon)L_1(1-\eta)\delta T)\right\| \\
&\le (\frac{\beta\varepsilon_3}{N} + \mu)\|L-L_1\| \\
&= \gamma_2\|L-L_1\|.
\end{aligned} \tag{3.7}
$$

If we do exact same thing for A_3, A_4, A_5 they also satisfied the Lipschitz condition. And they are contractions with $\gamma_i < 1, i = 1, \dots, 5$. This completes the proof.

By using kernels A_i and taking all initial values equal zero we rewrite the system given by equation (3.3). Then we define recursive formulas of this new system. Furthermore, we consider the differences and by taking the norm both sides of difference equations, we have,

$$
\begin{aligned}
\|(S_{n+1}-S_n)(t)\| &= \frac{2(1-a)}{2M(a)-aM(a)}\|A_1(t,S_n(t))-A_1(t,S_{n-1}(t))\| \\
&\quad+\frac{2a}{2M(a)-aM(a)}\int_0^t \|A_1(y,S_n(y))-A_1(y,S_{n-1}(y))\|dy \\
\|(L_{n+1}-L_n)(t)\| &= \frac{2(1-a)}{2M(a)-aM(a)}\|A_2(t,L_n(t))-A_2(t,L_{n-1}(t))\| \\
&\quad+\frac{2a}{2M(a)-aM(a)}\int_0^t \|A_2(y,L_n(y))-A_2(y,L_{n-1}(y))\|dy \\
\|(I_{n+1}-I_n)(t)\| &= \frac{2(1-a)}{2M(a)-aM(a)}\|A_3(t,I_n(t))-A_3(t,I_{n-1}(t))\| \\
&\quad+\frac{2a}{2M(a)-aM(a)}\int_0^t \|A_3(y,I_n(y))-A_3(y,I_{n-1}(y))\|dy \\
\|(T_{n+1}-T_n)(t)\| &= \frac{2(1-a)}{2M(a)-aM(a)}\|A_4(t,T_n(t))-A_4(t,T_{n-1}(t))\| \\
&\quad+\frac{2a}{2M(a)-aM(a)}\int_0^t \|A_4(y,T_n(y))-A_4(y,T_{n-1}(y))\|dy \\
\|(R_{n+1}-R_n)(t)\| &= \frac{2(1-a)}{2M(a)-aM(a)}\|A_5(t,R_n(t))-A_5(t,R_{n-1}(t))\| \\
&\quad+\frac{2a}{2M(a)-aM(a)}\int_0^t \|A_5(y,R_n(y))-A_5(y,R_{n-1}(y))\|dy
\end{aligned}
\tag{3.8}
$$

Theorem 3.2. If δ satisfies the condition down below then there is a solution.

$$\delta = \max\{\gamma_i\} < 1,\ \ \mathrm{i} = 1,2,\dots,5 \tag{3.9}$$

Proof. We define the functions

$$
\begin{aligned}
E_{1n}(t) &= S_{n+1}(t)-S(t), \\
E_{2n}(t) &= L_{n+1}(t)-L(t), \\
E_{3n}(t) &= I_{n+1}(t)-I(t), \\
E_{4n}(t) &= T_{n+1}(t)-T(t), \\
E_{5n}(t) &= R_{n+1}(t)-R(t)
\end{aligned}
$$

Then, for $E_{1n}(t)$, we get

$$\begin{aligned}\|E_{1n}(t)\| &\le \frac{2(1-a)}{2M(a)-aM(a)}\|A_1(t,S_n(t))-A_1(t,S_{n-1}(t))\| \\ &+\frac{2a}{2M(a)-aM(a)}\int_0^t\|A_1(y,S_n(y))-A_1(y,S_{n-1}(y))\|dy \\ &\le\left(\frac{2(1-a)}{2M(a)-aM(a)}+\frac{2a}{2M(a)-aM(a)}\right)\gamma_1\|S_n-S\| \\ &\le\left(\frac{2(1-a)}{2M(a)-aM(a)}+\frac{2a}{2M(a)-aM(a)}\right)^n\delta^n\|S-S_1\|.\end{aligned} \tag{3.10}$$

Using the same technique, we find

$$\begin{aligned}\|E_{2n}(t)\| &\le\left(\frac{2(1-a)}{2M(a)-aM(a)}+\frac{2a}{2M(a)-aM(a)}\right)^n\delta^n\|L-L_1\|, \\ \|E_{3n}(t)\| &\le\left(\frac{2(1-a)}{2M(a)-aM(a)}+\frac{2a}{2M(a)-aM(a)}\right)^n\delta^n\|I-I_1\|, \\ \|E_{4n}(t)\| &\le\left(\frac{2(1-a)}{2M(a)-aM(a)}+\frac{2a}{2M(a)-aM(a)}\right)^n\delta^n\|T-T_1\|, \\ \|E_{5n}(t)\| &\le\left(\frac{2(1-a)}{2M(a)-aM(a)}+\frac{2a}{2M(a)-aM(a)}\right)^n\delta^n\|R-R_1\|.\end{aligned} \tag{3.11}$$

Thus, from the above five functions, we find $E_{in}(t)\to 0,\ i=1,2,..5\ as\ n\to\infty\ for\ \delta<1$, which completes the proof.

In this part , we prove that our model has unique solution.

Theorem 3.3. The Caputo-Fabrizio fractional model (3.2) has a unique solution provided that the restrictions given by (3.11) hold true:

$$\left(\frac{2(1-a)}{2M(a)-aM(a)}+\frac{2a}{2M(a)-M(a)}\right)\gamma_i\le 1,\ \ i=1,2,\dots 5. \tag{3.11}$$

Proof. We assume each equation has two solutions such as S(t), L(t), I(t), T(t), R(t) and $\tilde{S}(t),\tilde{L}(t),\tilde{I}(t),\tilde{T}(t),\tilde{R}(t)$. Then we can write,

$$
\begin{aligned}
\tilde{S}(t) &= \frac{2(1-a)}{2M(a)-aM(a)}A_1(t,\tilde{S}(t)) + \frac{2a}{2M(a)-aM(a)}\int_0^t A_1(y,\tilde{S}(y))dy, \\
\tilde{L}(t) &= \frac{2(1-a)}{2M(a)-aM(a)}A_2(t,\tilde{L}(t)) + \frac{2a}{2M(a)-aM(a)}\int_0^t A_2(y,\tilde{L}(y))dy, \\
\tilde{I}(t) &= \frac{2(1-a)}{2M(a)-aM(a)}A_3(t,\tilde{I}(t)) + \frac{2a}{2M(a)-aM(a)}\int_0^t A_3(y,\tilde{I}(y))dy, \\
\tilde{T}(t) &= \frac{2(1-a)}{2M(a)-aM(a)}A_4(t,\tilde{T}(t)) + \frac{2a}{2M(a)-aM(a)}\int_0^t A_4(y,\tilde{T}(y))dy, \\
\tilde{R}(t) &= \frac{2(1-a)}{2M(a)-aM(a)}A_5(t,\tilde{R}(t)) + \frac{2a}{2M(a)-aM(a)}\int_0^t A_5(y,\tilde{R}(y))dy
\end{aligned}
\tag{3.12}
$$

Then, by using these equations and Theorem (3.1), we take the norm

$$
\begin{aligned}
\left\|S(t)-\tilde{S}(t)\right\| &\le \frac{2(1-a)}{2M(a)-aM(a)}\left\|A_1(t,S(t))-A_1(t,\tilde{S}(t))\right\| \\
&\quad + \frac{2a}{2M(a)-aM(a)}\int_0^t \left\|A_1(y,S(y))-A_1(y,\tilde{S}(y))\right\|dy \\
&\le \frac{2(1-a)}{2M(a)-aM(a)}\gamma_1\left\|S-\tilde{S}\right\| + \frac{2a\gamma_1}{2M(a)-aM(a)}\left\|S-\tilde{S}\right\|.
\end{aligned}
\tag{3.13}
$$

Which implies

$$
\left(\frac{2(1-a)}{2M(a)-aM(a)}\gamma_1 + \frac{2a\gamma_1}{2M(a)-aM(a)} - 1\right)\left\|S-\tilde{S}\right\| \ge 0. \tag{3.14}
$$

by condition (3.11), the inequality (3.14) is true provided that $\|S-\tilde{S}\| = 0$. Similarly, we use the same processes to prove that $L(t) = \tilde{L}(t),\ I(t) = \tilde{I}(t),\ T(t) = \tilde{T}(t),\ R(t) = \tilde{R}(t)$. Thus, the model has a unique solution.

REFERENCES

Caputo, M. (1967). Linear models of dissipation whose Q is almost frequency independent—II. Geophysical Journal International, 13(5), 529-539.

Caputo, M., & Fabrizio, M. (2015). A new definition of fractional derivative without singular kernel. Progr. Fract. Differ. Appl, 1(2), 1-13.

Losada, J., & Nieto, J. J. (2015). Properties of a new fractional derivative without singular kernel. Progr. Fract. Differ. Appl, 1(2), 87-92.

Ullah S, Khan MA,Farooq M. (2018), A fractional model for the dynamics of TB virus. Chaos,Solitons &Fractals.116,63-71.

Zhang, S.Q. (2000), The existence of a positive solution for a nonlinear fractional differential equation J. Math. Anal. Appl. 252.

New Approaches for Differentiable s-Convex Functions in The Fourth Sense via Fractional Integral Operators

Ahmet Ocak AKDEMİR[1], Merve NAZİK[2], Erhan SET[3] and Alper EKİNCİ[4]

[1] *Department of Mathematics, Faculty of Science and Arts,*
Ağrı İbrahim Çeçen University, Ağrı-TURKEY
[2] *Department of Mathematics, Faculty of Science and Arts,*
Ağrı İbrahim Çeçen University, Ağrı-TURKEY
[3] *Department of Mathematics, Faculty of Science and Arts,*
Ordu University, Ordu-TURKEY
[4] *Bandırma Vocational High School,*
Bandırma Onyedi Eylül University, Balıkesir-TURKEY

aocakakdemir@gmail.com
mervekosar92@hotmail.com
erhanset@yahoo.com
alperekinci@hotmail.com

ABSTRACT

The primary driving force behind this study is to connect the topic of inequalities with fractional integral operators, which are drawing interest due to their characteristics and widespread use. A new version of the Hermite-Hadamard (HH-) inequality is obtained for s-convex functions in the fourth sense for this purpose after certain fundamental notions are introduced. Numerous HH-type integral inequalities are found for the functions whose absolute values of the second derivatives are s-convex and s-concave using this integral equation that incorporates fractional integral operators with Mittag-Leffler kernel. The proof of the conclusions takes into consideration some well-known inequalities and hypothesis conditions, including Hölder's inequality and Young's inequality.

INTRODUCTION

With the help of researchers throughout many years, mathematics essentially began as a theoretical field with the goal of formulating events and occurrences in a variety of fields, such as physics, engineering, modeling, and mathematical biology, into a form that can be calculated. It has never been satisfied with this and is constantly searching for new and improved answers to issues. One of the key methods used by mathematics to solve problems in the real world is fractional analysis. Recent research has actually demonstrated that fractional analysis accomplishes this goal better than classical analysis. The fundamental tenet of fractional analysis is the introduction of novel fractional derivatives and integral operators, followed by an investigation of the benefits of each operator using examples from real-world problems, modeling studies, and comparisons. In an effort to advance fractional analysis and introduce the most efficient operators to the literature, new fractional derivatives and associated integral operators have been developed. Fractional operators differ in this dynamic process because of various characteristics of kernel structures, the time memory effect, and the need to obtain general forms.

Definition 1 *(see[1]) Let* $f \in L[\varepsilon_1, \varepsilon_2]$. *The Riemann-Liouville integrals* $J^{\xi}_{\varepsilon_1^+} f$ *and* $J^{\xi}_{\varepsilon_2^-} f$ *of order* $\xi > 0$ *with* $\varepsilon_1, \varepsilon_2 \geq 0$ *are defined by*

$$\left(J^{\xi}_{\varepsilon_1^+}\right) f(y) = \frac{1}{\Gamma(\xi)} \int_{\varepsilon_1}^{y} (y-t)^{\xi-1} f(t)dt; \qquad y > \varepsilon_1, \tag{1.1}$$

and

$$\left(J^{\xi}_{\varepsilon_2^-}\right) f(y) = \frac{1}{\Gamma(\xi)} \int_{y}^{\varepsilon_2} (t-y)^{\xi-1} f(t)dt; \qquad y < \varepsilon_2, \tag{1.2}$$

respectively, where $\Gamma(.)$ is the gamma function and $\left(J^{0}_{\varepsilon_1^+}\right) f(y) = \left(J^{0}_{\varepsilon_2^-}\right) f(y) = f(y)$.

Researchers in both mathematical analysis and applied mathematics have used the Riemann-Liouville fractional integral operator to solve a variety of issues (see [2]-[4]).The most well-known operators in fractional analysis for a long time were the Caputo and Caputo-Fabrizio derivatives and the Riemann-Liouville integrals.

Definition 2 *[9]. Let* $f \in \mathrm{H}^1(0,\Theta_2), \Theta_2 > \Theta_1, \alpha \in [0,1]$, *then the definition of the new Caputo fractional derivative is*

$$^{CF}\mathrm{D}f(\Xi) = \frac{M(\alpha)}{1-\alpha} \int_{K}^{\Xi} f'(s) \exp\left[-\frac{\alpha}{(1-\alpha)}(\Xi - s)\right] ds, \tag{1.3}$$

where $M(\alpha)$ is normalization function.

Definition 3 *[10] Let* $f \in H^1(0,\Theta_2), \Theta_2 > \Theta_1, \alpha \in [0,1]$, *then the definition of the left and right side of Caputo-Fabrizio fractional integral is*

$$\left({}^{CF}_{\Theta_1}\mathrm{I}^{\alpha}\right)(\Xi) = \frac{1-\alpha}{M(\alpha)} f(\Xi) + \frac{\alpha}{M(\alpha)} \int_{\Theta_1}^{\Xi} f(y)dy, \tag{1.4}$$

and

$$\left({}^{CF}\mathrm{I}^{\alpha}_{\Theta_2}\right)(\Xi) = \frac{1-\alpha}{M(\alpha)} f(\Xi) + \frac{\alpha}{M(\alpha)} \int_{\Xi}^{\Theta_2} f(y)dy, \tag{1.5}$$

where $B(\alpha)$ is the normalization function.

The authors' definition of the Caputo-Fabrizio fractional integral operator is based on this intriguing fractional derivative operator. Despite being a highly functional operator by definition, the Caputo-Fabrizio fractional derivative, which is employed in dynamical systems, physical phenomena, disease models, and many other domains, has a weakness in that it does not satisfy the initial requirements in the exceptional case $\alpha = 1$. The new derivative operator created by Atangana-Baleanu, which contains versions in the sense of Caputo and Riemann, has provided the improvement to remove this flaw. The normalization function will be denoted by B(α) in the follow-up to this study and share the same qualities as the M(α) defined by Caputo-Fabrizio.

Definition 4 *[11] Let* $f \in H^1(\Theta_1, \Theta_2), \Theta_2 > \Theta_1, \alpha \in [0,1]$, *then the definition of the new fractional derivative is given:*

$$ {}^{ABC}_{\Theta_1}\mathrm{D}^{\alpha}_{\Xi}[f(\Xi)] = \frac{B(\alpha)}{1-\alpha} \int_{\Theta_1}^{\Xi} f'(x) E_{\alpha}\left[-\alpha \frac{(\Xi - x)^{\alpha}}{1-\alpha}\right] dx. \tag{1.6} $$

Definition 5 *Let* $f \in H^1(\Theta_1, \Theta_2), \Theta_2 > \Theta_1, \alpha \in [0,1]$, *then the definition of the new fractional derivative is given:*

$$ {}^{ABR}_{\Theta_1}\mathrm{D}^{\alpha}_{\Xi}[f(\Xi)] = \frac{B(\alpha)}{1-\alpha} \frac{d}{d\Xi} \int_{\Theta_1}^{\Xi} f(x) E_{\alpha}\left[-\alpha \frac{(\Xi - x)^{\alpha}}{1-\alpha}\right] dx. \tag{1.7} $$

The kernel of Atangana-Balenau derivatives seen above is nonlocal. In equation (1.7), when the function is constant, we get zero.

After these definitions, Atangana Balenau also defined the fractional integral operator.

Definition 6 *[12] The fractional integral associate to the new fractional derivative with nonlocal kernel of a function* $f \in H^1(\Theta_1, \Theta_2)$ *as defined:*

$$ {}^{AB}_{\Theta_1}I^{\alpha}_{\Xi}\{f(\Xi)\} = \frac{1-\alpha}{B(\alpha)} f(\Xi) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_{\Theta_1}^{\Xi} f(y)(\Xi - y)^{\alpha-1} dy, \tag{1.8} $$

where $\Theta_2 > \Theta_1$ and $\alpha \in [0,1]$.

In [13], the right-hand side of the Abdeljawad and Baleanu integral operator is calculated. The right fractional new integral with ML kernel of order $\alpha \in [0,1]$ is defined by

$$ \left({}^{AB}I^{\alpha}_{\Theta_2}\right)\{f(\Xi)\} = \frac{1-\alpha}{B(\alpha)} f(\Xi) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_{\Xi}^{\Theta_2} f(y)(y - \Xi)^{\alpha-1} dy. \tag{1.9} $$

Gamma function $\Gamma(\alpha)$ is used here. The fractional Atangana-Baleanu integral of a positive function is positive because $B(\alpha) > 0$ the so-called normalization function gives rise to this result. It should be observed that we regain the common integral when the order $\alpha \to 1$. Additionally, whenever the fractional order $\alpha \to 0$, the original function is restored.

Following a brief introduction to fractional analysis, we will go over some fundamental ideas related to convex functions and inequalities. Let's review the s-convex function in the second sense, the s-convex function in the fourth sense, and the HH inequality to refresh our memories.

Definition 7 *[6]. The function* $f:[0,\infty)\to\mathbb{R}$ *is said to be* $s-$*convex in the second sense if for every* $x,y\in 0,\infty)$ *and* $t\in 0,1]$ *and for some fixed* $s\in(0,1]$ *we have:*

$$f(tx+(1-t)y)\le t^s f(x)+(1-t)^s f(y).$$

The class of $s-$convex functions in the second sense is usually denoted by K_s^2.

If we choose $s=1$, it can be easily seen that $s-$convexity of functions defined on $x,y\in 0,\infty)$. If $s\in(0,1)$, $f\in K_s^2$ implies $f([0,\infty))\subseteq[0,\infty)$, i.e., this has been proven for all functions from $K_s^2, s\in(0,1)$, are nonnegative.

Definition 8 *[8] Let* $U\subseteq\mathbb{R}^n$ *be a convex set and let* $s\in(0,1]$ *and* $f:U\to\mathbb{R}.f$ *is said to be* $s-$*convex function in the fourth sense if*

$$f(tx+(1-t)y)\le t^{\frac{1}{s}} f(x)+(1-t)^{\frac{1}{s}} f(y)$$

for all $x,y\in U$ and $t\in[0,1]$.

The class of $s-$convex functions in the fourth sense is denoted by K_s^4.

With its various modifications, refinements, and iterations, the well-known HH-inequality, which is based on convex functions, generates lower and upper limits for the mean value in the Cauchy sense and is presented as follows:

Assume that $f:I\subset\mathbb{R}\to\mathbb{R}$ is a convex mapping on $I\subseteq\mathbb{R}$, where $\varepsilon_1,\varepsilon_2\in I$, with $\varepsilon_1<\varepsilon_2$. The HH-inequality for convex mappings can be presented as follows (see [14]):

$$f\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)\le\frac{1}{\varepsilon_2-\varepsilon_1}\int_{\varepsilon_1}^{\varepsilon_2} f(t)dt\le\frac{f(\varepsilon_1)+f(\varepsilon_2)}{2}. \tag{1.10}$$

Dragomir and Fitzpatrick have carried out a novel HH-inequality for s-convex maps in the second sense in [16].

Theorem 1 *(see [24]). Assume that* $f:[0,\infty)\to[0,\infty)$ *is a s-convex function in the fourth sense, where* $s\in(0,1)$, *and let* $\varepsilon_1,\varepsilon_2\in[0,\infty),\varepsilon_1<\varepsilon_2$. *If* $f\in L[\varepsilon_1,\varepsilon_2]$, *then one has the following:*

$$2^{\frac{1}{s}-1} f\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)\le\frac{1}{\varepsilon_2-\varepsilon_1}\int_{\varepsilon_1}^{\varepsilon_2} f(t)dt\le\frac{s}{s+1}[f(\varepsilon_1)+f(\varepsilon_2)]. \tag{1.11}$$

Here, we must note that $k=\dfrac{s}{s+1}$ is the best possible constant in (1.11).

We recommend reading the works ([14]-[22]) for more information on the various convex functions and generalizations, novel variations, and various manifestations of this significant double inequality.

The structure of this study is as follows. Prior to anything else, the fundamental ideas that would be applied in the study were determined, and the infrastructure needed to support science was built. In the second sense, Atangana-Baleanu integral operators for s-convex in the second sense were shown in [23]. Atangana-Baleanu integral operators for s-convex functions in the fourth sense are discovered, leading to a new generalization of the HH-inequality in the main findings section.

RESULTS

We begin this part by presenting the following inequalities, which use novel fractional integral operators developed by Atangana and Baleanu to represent variations of the HH-inequality for s-convex maps in the fourth sense. The functions $\Gamma(\alpha), B(\alpha) > 0$, and β_x are referred to as the gamma function, normalization function, and incomplete beta function, respectively, throughout the study.

Theorem 2 *Let* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *be an s-convex function in the fourth sense,* $s \in (0,1]$, *and* $a, b \in \mathbb{R}^+$ *with* $a < b$. *If* $f \in L[a,b]$, *the inequalities for Atangana-Baleanu integral operators for all* $\alpha \in (0,1]$ *are obtained as follows:*

$$\begin{aligned} & 2^{\frac{1}{s}} \frac{f\left(\frac{a+b}{2}\right)}{B(\alpha)\Gamma(\alpha)} + \frac{1-\alpha}{(b-a)^{\alpha}} \left[\frac{f(a)+f(b)}{B(\alpha)} \right] \\ & \leq \frac{1}{(b-a)^{\alpha}} \left[{}^{AB}_{a}I^{\alpha}_{b}\{f(b)\} + {}^{AB}I^{\alpha}_{b}\{f(a)\} \right] \\ & \leq \left[\frac{f(a)+f(b)}{B(\alpha)} \right] \left[\frac{\alpha}{\Gamma(\alpha)(\alpha+\frac{1}{s})} + \frac{1-\alpha}{(b-a)^{\alpha}} + \frac{\alpha\beta(\alpha, 1+\frac{1}{s})}{\Gamma(\alpha)} \right]. \end{aligned} \tag{2.1}$$

Proof. As f is an s-convex function in the fourth sense, we can write

$$f(ta+(1-t)b) \leq t^{\frac{1}{s}} f(a) + (1-t)^{\frac{1}{s}} f(b)$$

for all $t \in [0,1]$. Multiplying the above inequality with $t^{\alpha-1}$ and then integrating the obtained inequality on $[0,1]$, we have

$$\begin{aligned} & \int_0^1 t^{\alpha-1} f(ta+(1-t)b)dt \\ & \leq \left[f(a) \int_0^1 t^{\alpha+\frac{1}{s}-1} dt + f(b) \int_0^1 t^{\alpha-1} (1-t)^{\frac{1}{s}} dt \right]. \end{aligned}$$

If we multiply both sides of the last inequality by $\frac{\alpha(b-a)^{\alpha}}{B(\alpha)\Gamma(\alpha)}$, and then if we add the term $\frac{1-\alpha}{B(\alpha)} f(b)$, we get

$$\frac{\alpha(b-a)^{\alpha}}{B(\alpha)\Gamma(\alpha)}\int_0^1 t^{\alpha-1} f(ta+(1-t)b)dt+\frac{1-\alpha}{B(\alpha)} f(b)$$

$$\leq \frac{\alpha(b-a)^{\alpha}}{B(\alpha)\Gamma(\alpha)}\left[f(a)\int_0^1 t^{\alpha+\frac{1}{s}-1} dt+ f(b)\int_0^1 t^{\alpha-1}(1-t)^{\frac{1}{s}} dt\right]+\frac{1-\alpha}{B(\alpha)} f(b).$$

By making use of the change of variable $ta+(1-t)b=y$, we have

$$\begin{aligned} &{}_{a}^{AB}I_b^{\alpha}\{f(b)\} \\ &\leq f(b)\left[\frac{1-\alpha}{B(\alpha)}+\frac{\alpha(b-a)^{\alpha}\beta(\alpha,\frac{1}{s}+1)}{B(\alpha)\Gamma(\alpha)}\right]+\frac{\alpha(b-a)^{\alpha}}{B(\alpha)\Gamma(\alpha)(\alpha+\frac{1}{s})} f(a). \end{aligned} \tag{2.2}$$

and similary we get

$$\begin{aligned} &{}^{AB}I_b^{\alpha}\{f(a)\} \\ &\leq f(a)\left[\frac{1-\alpha}{B(\alpha)}+\frac{\alpha(b-a)^{\alpha}\beta(\alpha,\frac{1}{s}+1)}{B(\alpha)\Gamma(\alpha)}\right]+\frac{\alpha(b-a)^{\alpha}}{B(\alpha)\Gamma(\alpha)(\alpha+\frac{1}{s})} f(b). \end{aligned} \tag{2.3}$$

If we take into account the disparities in (2.2) and (2.3), we see that the second inequality in (2.1).

We employ the fact that, for all $u,v\in \mathrm{R}^{+}$ in order to derive the first inequality in (2.1), we have

$$f\left(\frac{u+v}{2}\right)\leq \frac{f(u)+f(v)}{2^{\frac{1}{s}}}. \tag{2.4}$$

Now, let $u=ta+(1-t)b$ and $v=(1-t)a+tb$ with $t\in[0,1]$. Then we get by (2.4) that

$$f\left(\frac{a+b}{2}\right)\leq \frac{f(ta+(1-t)b)+f((1-t)a+tb)}{2^{\frac{1}{s}}}.$$

Multiplying the above inequality with $t^{\alpha-1}$ and then integrating this inequality on [0,1], we have

$$2^{\frac{1}{s}}\frac{f\left(\frac{a+b}{2}\right)}{\alpha}$$

$$\leq \int_0^1 t^{\alpha-1} f(ta+(1-t)b)dt+\int_0^1 t^{\alpha-1} f((1-t)a+tb)dt.$$

If we multiply both sides of the last inequality $\frac{\alpha(b-a)^{\alpha}}{B(\alpha)\Gamma(\alpha)}$ and then if we add the term $\frac{1-\alpha}{B(\alpha)}[f(a)+f(b)]$ to two sides of the resulting inequality, we get

$$\begin{aligned}
&2^{\frac{1}{s}}\frac{(b-a)^{\alpha}}{B(\alpha)\Gamma(\alpha)} f\left(\frac{a+b}{2}\right)+\frac{1-\alpha}{B(\alpha)}[f(a)+f(b)] \\
&\leq \frac{\alpha(b-a)^{\alpha}}{B(\alpha)\Gamma(\alpha)}\int_0^1 t^{\alpha-1} f(ta+(1-t)b)dt \\
&\quad+\frac{\alpha(b-a)^{\alpha}}{B(\alpha)\Gamma(\alpha)}\int_0^1 t^{\alpha-1} f((1-t)a+tb)dt \\
&\quad+\frac{1-\alpha}{B(\alpha)}[f(a)+f(b)].
\end{aligned}$$

The change of variables $ta+(1-t)b=y$ and $tb+(1-t)a=z$ gives us

$$\begin{aligned}
&2^{\frac{1}{s}}\frac{(b-a)^{\alpha}}{B(\alpha)\Gamma(\alpha)} f\left(\frac{a+b}{2}\right)+\frac{1-\alpha}{B(\alpha)}[f(a)+f(b)] \\
&\leq \left[{}^{AB}_{a}I^{\alpha}_{b}\{f(b)\}+{}^{AB}I^{\alpha}_{b}\{f(a)\}\right]
\end{aligned} \tag{2.5}$$

If we multiply both sides of (2.5) by $\frac{1}{(b-a)^{\alpha}}$, we get the first inequality in (2.1).

As we move on in this part, we present an equality for integral operators for Atangana-Baleanu that has second order derivatives.

Lemma 1 *Let $a<b, a,b\in I^0$ and $f: I\subset \mathrm{R}\to\mathrm{R}$ be a differentiable function on I^0. If $f''\in[a,b]$, the identity for Atangana- Baleanu integral operators in equation (2.6) is valid for all $\alpha\in(0,1]$:*

$$\begin{aligned}
&\frac{1}{b-a}\left[({}^{AB}I^{\alpha}_{\frac{a+b}{2}})\{f(a)\}+{}^{AB}_{\frac{a+b}{2}}I^{\alpha}_{b}\{f(b)\}\right] \\
&\quad-\frac{1-\alpha}{(b-a)B(\alpha)}[f(a)+f(b)] \\
&\quad-\frac{(b-a)^{\alpha-1}}{2^{\alpha-1}B(\alpha)\Gamma(\alpha)}f\left(\frac{a+b}{2}\right) \\
&=\frac{(b-a)^{\alpha+1}}{2(\alpha+1)B(\alpha)\Gamma(\alpha)} \\
&\qquad\times\int_0^1 m^{\alpha}(t)[f''(ta+(1-t)b)+f''(tb+(1-t)a)],
\end{aligned} \tag{2.6}$$

where

$$m^{\alpha}(t)=\begin{cases} t^{\alpha+1}, & t\in[0,\frac{1}{2}), \\ (1-t)^{\alpha+1}, & t\in[\frac{1}{2},1], \end{cases}$$

and also $\Gamma(\alpha)$ is a gamma function and $B(\alpha)>0$.

The proof for the aforementioned lemma may be found in [23].

Now, using the new integral equation and the s-convexity identity, we will create generalizations of the HH-type inequalities for Atangana-Baleanu fractional integral operators. The following terms are indicated with a F throughout the study:

$$\begin{aligned}
F=&\frac{1}{b-a}\left[({}^{AB}I^{\alpha}_{\frac{a+b}{2}})\{f(a)\}+{}^{AB}_{\frac{a+b}{2}}I^{\alpha}_{b}\{f(b)\}\right] \\
&-\frac{1-\alpha}{(b-a)B(\alpha)}[f(a)+f(b)]-\frac{(b-a)^{\alpha-1}}{2^{\alpha-1}B(\alpha)\Gamma(\alpha)}f\left(\frac{a+b}{2}\right).
\end{aligned}$$

Theorem 3 *Let* $a<b, a,b\in I^0$ *and* $f:I\subset[0,\infty)\to\mathbb{R}$ *be a differentiable function on* I^0 *and* $f''\in L(a,b)$. *If* $|f''|$ *is an s-convex function in the fourth sense on* $[a,b]$ *for some fixed* $s\in(0,1]$, *we obtain the following inequality for Atangana-Baleanu integral operators:*

$$|F|\leq\frac{(b-a)^{\alpha+1}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(\frac{(\frac{1}{2})^{\alpha+\frac{1}{s}+2}}{\alpha+\frac{1}{s}+2}+\beta_{\frac{1}{2}}(\alpha+2,\frac{1}{s}+1)\right)(|f''(a)|+|f''(b)|), \tag{2.7}$$

where $\alpha\in(0,1]$.

Proof. By using the equality in (2.6) and the s-convexity of $|f''|$, we have

$$
\begin{aligned}
|F| &\le \frac{(b-a)^{\alpha+1}}{2(\alpha+1)B(\alpha)\Gamma(\alpha)} \\
&\quad \times \int_0^1 |m^{\alpha}(t)| \left[|f''(ta+(1-t)b)| + |f''(tb+(1-t)a)| \right] dt \\
&\le \frac{(b-a)^{\alpha+1}}{2(\alpha+1)B(\alpha)\Gamma(\alpha)} \left\{ \int_0^{\frac{1}{2}} t^{\alpha+1} \left[t^{\frac{1}{s}} |f''(a)| + (1-t)^{\frac{1}{s}} |f''(b)| \right] dt \right. \\
&\quad + \int_{\frac{1}{2}}^1 (1-t)^{\alpha+1} \left[t^{\frac{1}{s}} |f''(a)| + (1-t)^{\frac{1}{s}} |f''(b)| \right] dt \\
&\quad + \int_0^{\frac{1}{2}} t^{\alpha+1} \left[t^{\frac{1}{s}} |f''(b)| + (1-t)^{\frac{1}{s}} |f''(a)| \right] dt \\
&\quad \left. + \int_{\frac{1}{2}}^1 (1-t)^{\alpha+1} \left[t^{\frac{1}{s}} |f''(b)| + (1-t)^{\frac{1}{s}} |f''(a)| \right] dt \right\}.
\end{aligned}
$$

Then, after obtaining the calculations required, we finish the inequality's proof in (2.7).

Corollary 1 *In Theorem 3, if we choose* $s=1$, *we have the following inequality:*

$$
|F| \le \frac{(b-a)^{\alpha+1}}{(\alpha+1)B(\alpha)\Gamma(\alpha)} \left(\frac{\frac{1}{2}^{\alpha+3}}{\alpha+3} + \beta_{\frac{1}{2}}(\alpha+2,2) \right) (|f''(a)| + |f''(b)|).
$$

Corollary 2 *In Theorem 3, if* $|f''| \le M$ *on* $I^0, M>0$, *we have the following inequality:*

$$
|F| \le \frac{2M(b-a)^{\alpha+1}}{(\alpha+1)B(\alpha)\Gamma(\alpha)} \left(\frac{\frac{1}{2}^{\alpha+\frac{1}{s}+2}}{\alpha+\frac{1}{s}+2} + \beta_{\frac{1}{2}}(\alpha+2,\frac{1}{s}+1) \right).
$$

Theorem 4 *Let* $a<b, a,b \in I^0$ *and* $f: I \subset [0,\infty) \to \mathbb{R}$ *be a differentiable mapping on* I^0 *and* $f'' \in L[a,b]$. *If* $|f''|^q$ *is an s-convex function in the fourth sense on* $[a,b]$ *for some fixed* $s \in (0,1]$, *we obtain the following inequality for Atangana-Baleanu integral operators:*

$$
|F| \le \frac{(b-a)^{\alpha+1}}{(\alpha+1)B(\alpha)\Gamma(\alpha)} \left(\frac{(\frac{1}{2})^{\alpha p+p}}{\alpha p+p+1} \right)^{\frac{1}{p}} \frac{1}{(\frac{1}{s}+1)^{\frac{1}{q}}} (|f''(a)| + |f''(b)|), \tag{2.8}
$$

where $\alpha \in (0,1], q>1,$ and $\frac{1}{p}+\frac{1}{q}=1.$

Proof. By using the equality in (2.6) and Hölder's inequality, we get

$$
\begin{aligned}
|F| &\leq \frac{(b-a)^{\alpha+1}}{2(\alpha+1)B(\alpha)\Gamma(\alpha)} \\
&\quad \times \int_0^1 |m^{\alpha}(t)|[|f''(ta+(1-t)b)|+|f''(tb+(1-t)a)|]dt \\
&\leq \frac{(b-a)^{\alpha+1}}{2(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(\int_0^1 |m^{\alpha}(t)|^p\, dt\right)^{\frac{1}{p}} \\
&\quad \times \left[\left(\int_0^1 |f''(ta+(1-t)b)|^q\, dt\right)^{\frac{1}{q}}+\left(\int_0^1 |f''(tb+(1-t)a)|^q\, dt\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

We apply the fourth meaning of the s-convexity on $[a,b]$ to arrive at the result, and we then make use of the fact that

$$\sum_{k=1}^{n}(u_k+v_k)^m \leq \sum_{k=1}^{n}u_k^m+\sum_{k=1}^{n}v_k^m$$

for $0\leq m<1, u_1,u_2,\ldots,u_n \geq 0, v_1,v_2,\ldots,v_n \geq 0$.

So, we obtained the inequality (2.8). The proof is completed.

Corollary 3 *In Theorem 4, if we choose* $s=1$, *we have following inequality:*

$$|F| \leq \frac{(b-a)^{\alpha+1}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(\frac{(\frac{1}{2})^{\alpha p+p}}{\alpha p+p+1}\right)^{\frac{1}{p}}\frac{1}{2^{\frac{1}{q}}}(|f''(a)|+|f''(b)|).$$

Corollary 4 *In Theorem 4, if* $|f''|\leq M$ *on* $I^0, M>0$*,we have the following inequality:*

$$|F| \leq \frac{2M(b-a)^{\alpha+1}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(\frac{(\frac{1}{2})^{\alpha p+p}}{\alpha p+p+1}\right)^{\frac{1}{p}}\frac{1}{(\frac{1}{s}+1)^{\frac{1}{q}}}.$$

Theorem 5 *According to Theorem 4's assumptions, the inequality in (2.9) results:*

$$
\begin{aligned}
|F| \ \le & \frac{(b-a)^{\alpha+1}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(\frac{1}{2^{\alpha+1}(\alpha+2)}\right)^{\frac{1}{p}} \\
& \times\left(\frac{(\frac{1}{2})^{\alpha+\frac{1}{s}+2}}{\alpha+\frac{1}{s}+2}+\beta_{\frac{1}{2}}(\alpha+2,\frac{1}{s}+1)\right)^{\frac{1}{q}}(|f''(a)|^q+|f''(b)|^q)^{\frac{1}{q}},
\end{aligned} \tag{2.9}
$$

where $\alpha\in(0,1], q>1, \frac{1}{p}+\frac{1}{q}=1.$

Proof. When we use Hölder inequality from a different point of view, we can write

$$
\begin{aligned}
|F| \ \le & \frac{(b-a)^{\alpha+1}}{2(\alpha+1)B(\alpha)\Gamma(\alpha)} \\
& \times\int_0^1|m^\alpha(t)|[|f''(ta+(1-t)b)|+|f''(tb+(1-t)a)|]dt \\
\le & \frac{(b-a)^{\alpha+1}}{2(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(\int_0^1|m^\alpha(t)|^p\,dt\right)^{\frac{1}{p}} \\
& \times\left[\left(\int_0^1|f''(ta+(1-t)b)|^q\,dt\right)+\left(\int_0^1|f''(tb+(1-t)a)|^q\,dt\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

If we apply the s-convexity of $|f''|^q$ and calculate the above integrals, we get the desired.

Corollary 5 *In Theorem 5, if we choose* $s=1$, *we have the following inequality:*

$$
\begin{aligned}
|F| \le & \frac{(b-a)^{\alpha+1}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(\frac{1}{2^{\alpha+1}(\alpha+2)}\right)^{\frac{1}{p}} \\
& \times\left(\frac{(\frac{1}{2})^{\alpha+3}}{\alpha+3}+\beta_{\frac{1}{2}}(\alpha+2,2)\right)^{\frac{1}{q}}(|f''(a)|^q+|f''(b)|^q)^{\frac{1}{q}}.
\end{aligned}
$$

Corollary 6 *In Theorem 5, if* $|f''|\le M$ *on* $I^0, M>0$, *we have the following inequality:*

$$|F| \leq \frac{2^{\frac{1}{q}} M (b-a)^{\alpha+1}}{(\alpha+1)B(\alpha)\Gamma(\alpha)} \left(\frac{1}{2^{\alpha+1}(\alpha+2)} \right)^{\frac{1}{p}}$$
$$\times \left(\frac{(\frac{1}{2})^{\alpha+\frac{1}{s}+2}}{\alpha+\frac{1}{s}+2} + \beta_{\frac{1}{2}}(\alpha+2, \frac{1}{s}+1) \right)^{\frac{1}{q}}.$$

Theorem 6 *Let* $a<b, a,b \in I^0$ *and* $f: I \subset [0,\infty) \to \mathbb{R}$ *be a differentiable function on* I^0 *and* $f'' \in L[a,b]$. *If* $|f''|^q$ *is an s-convex function in the fourth sense on* $[a,b]$ *for some fixed* $s \in (0,1]$, *we obtain the following inequality in (2.10) for Atangana-Baleanu integral operators:*

$$|F| \leq \frac{(b-a)^{\alpha+1}}{(\alpha+1)B(\alpha)\Gamma(\alpha)} \left(\frac{(\frac{1}{2})^{(\alpha+1)(\frac{q-p}{q-1})}(q-1)}{(\alpha+1)(q-p)+q-1} \right)^{1-\frac{1}{q}} (|f''(a)|^q + |f''(b)|^q)^{\frac{1}{q}}$$
$$\times \left(\frac{(\frac{1}{2})^{(\alpha+1)p+\frac{1}{s}+1}}{(\alpha+1)p+\frac{1}{s}+1} + \beta_{\frac{1}{2}}((\alpha+1)p+1, \frac{1}{s}+1) \right)^{\frac{1}{q}}, \tag{2.10}$$

where $\alpha \in (0,1], q \geq p > 1$.

Proof. By using Hölder's inequality in a different way, we can write

$$|F| \leq \frac{(b-a)^{\alpha+1}}{2(\alpha+1)B(\alpha)\Gamma(\alpha)} \{ \left(\int_0^1 |m^\alpha(t)|^{\frac{q-p}{q-1}} dt \right)^{1-\frac{1}{q}}$$
$$\times \left(\int_0^1 |m^\alpha(t)|^p |ta+(1-t)b|^q dt \right)^{\frac{1}{q}}$$
$$+ \left(\int_0^1 |m^\alpha(t)|^{\frac{q-p}{q-1}} dt \right)^{1-\frac{1}{q}}$$
$$\times \left(\int_0^1 |m^\alpha(t)|^p |f''(tb+(1-t)a)|^q dt \right)^{\frac{1}{q}} \}.$$

If we use the s-convexity of $|f''|^q$ above, we have

$$
\begin{aligned}
|F| \leq & \frac{(b-a)^{\alpha+1}}{2(\alpha+1)B(\alpha)\Gamma(\alpha)}\{\left(\int_0^1 |m^\alpha(t)|^{\frac{q-p}{q-1}}\,dt\right)^{1-\frac{1}{q}} \\
& \times\left(\int_0^1 |m^\alpha(t)|^p\,[t^{\frac{1}{s}}\,|f''(a)|^q+(1-t)^{\frac{1}{s}}\,|f''(b)|^q]dt\right)^{\frac{1}{q}} \\
& +\frac{(b-a)^{\alpha+1}}{2(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(\int_0^1 |m^\alpha(t)|^{\frac{q-p}{q-1}}\,dt\right)^{1-\frac{1}{q}} \\
& \times\left(\int_0^1 |m^\alpha(t)|^p\,[t^{\frac{1}{s}}\,|f''(a)|^q+(1-t)^{\frac{1}{s}}\,|f''(b)|^q]dt\right)^{\frac{1}{q}}.
\end{aligned}
$$

By making the necessary integral calculations, the proof is completed.

Corollary 7 *In Theorem 6, if we choose* $s=1$*, we have the following inequality:*

$$
\begin{aligned}
|F| \leq & \frac{(b-a)^{\alpha+1}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(\frac{(\frac{1}{2})^{(\alpha+1)(\frac{q-p}{q-1})}(q-1)}{(\alpha+1)(q-p)+q-1}\right)^{1-\frac{1}{q}} \\
& \times(|f''(a)|^q+|f''(b)|^q)^{\frac{1}{q}}\left(\frac{(\frac{1}{2})^{(\alpha+1)p+2}}{(\alpha+1)p+2}+\beta_{\frac{1}{2}}((\alpha+1)p+1,2)\right)^{\frac{1}{q}}.
\end{aligned} \tag{2.11}
$$

Corollary 8 *In Theorem 6, if* $|f''|\leq M$ *on* $I^0, M>0$*, we have the following inequality:*

$$
\begin{aligned}
|F| \leq & \frac{2^{\frac{1}{q}}M(b-a)^{\alpha+1}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(\frac{(\frac{1}{2})^{(\alpha+1)(\frac{q-p}{q-1})}(q-1)}{(\alpha+1)(q-p)+q-1}\right)^{1-\frac{1}{q}} \\
& \times\left(\frac{(\frac{1}{2})^{(\alpha+1)p+\frac{1}{s}+1}}{(\alpha+1)p+\frac{1}{s}+1}+\beta_{\frac{1}{2}}((\alpha+1)p+1,\frac{1}{s}+1)\right)^{\frac{1}{q}},
\end{aligned}
$$

Theorem 7 *Let* $a<b, a,b\in I^0$ *and* $f: I\subset[0,\infty)\to \mathbb{R}$ *be a differentiable function on* I^0 *and* $f''\in L[a,b]$. *If* $|f''|^q$ *is an s-convex function in the fourth sense on* $[a,b]$ *for some fixed* $s\in(0,1]$, *we obtain the following inequality in (2.12) for Atangana-Baleanu integral operators:*

$$|F|\leq \frac{(b-a)^{\alpha+1}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(\frac{(\frac{1}{2})^{(\alpha+1)p}}{((\alpha+1)p+1)p}+\frac{|f''(a)|^q+|f''(b)|^q}{(\frac{1}{s}+1)q}\right), \tag{2.12}$$

where $\alpha\in(0,1], q>1$.

Proof. By using Lemma 1, we have

$$|F|\leq \frac{(b-a)^{\alpha+1}}{2(\alpha+1)B(\alpha)\Gamma(\alpha)} \times\int_0^1 |m^\alpha(t)|[|f''(ta+(1-t)b)|+|f''(tb+(1-t)a)|]dt.$$

By using Young's inequality as $xy\leq \frac{1}{p}x^p+\frac{1}{q}y^q$, we get

$$\begin{aligned}|F|\leq& \frac{(b-a)^{\alpha+1}}{2(\alpha+1)B(\alpha)\Gamma(\alpha)}\\ &\times\{\frac{1}{p}\int_0^1|m^\alpha(t)|^p\,dt+\frac{1}{q}\int_0^1|f''(ta+(1-t)b)|^q\,dt\\ &+\frac{1}{p}\int_0^1|m^\alpha(t)|^p\,dt+\frac{1}{q}\int_0^1|f''(tb+(1-t)a)|^q\,dt\}.\end{aligned}$$

By using the s-convexity of $|f''|^q$ and by simple calculations, we provide the result.

Corollary 9 *In Theorem 7, if we choose* $s=1$, *we have the following inequality:*

$$|F|\leq \frac{(b-a)^{\alpha+1}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(\frac{(\frac{1}{2})^{(\alpha+1)p}}{((\alpha+1)p+1)p}+\frac{|f''(a)|^q+|f''(b)|^q}{2q}\right).$$

Corollary 10 *In Theorem 7, if* $|f''|\leq M$ *on* $I^0, M>0$, *we have the following inequality:*

$$|F|\leq \frac{(b-a)^{\alpha+1}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(\frac{(\frac{1}{2})^{(\alpha+1)p}}{((\alpha+1)p+1)p}+\frac{2M^q}{(\frac{1}{s}+1)q}\right).$$

Theorem 8 *Let $a<b, a,b\in I^0$ and $f:I\subset[0,\infty)\to \mathbb{R}$ be a differentiable function on I^0 and $f''\in L[a,b]$. If $|f''|^q$ is an s-concave function in the fourth sense on $[a,b]$ for some fixed $s\in(0,1]$, we obtain the following inequality in (2.13) for Atangana-Baleanu integral operators:*

$$|F|\leq \frac{(b-a)^{\alpha+1}}{(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(\frac{(\frac{1}{2})^{(\alpha+1)p}}{((\alpha+1)p+1)}\right)^{\frac{1}{p}} 2^{\frac{\frac{1}{s}-1}{q}}\left|f''\left(\frac{a+b}{2}\right)\right|, \tag{2.13}$$

where $\alpha\in(0,1], q>1, \frac{1}{p}+\frac{1}{q}=1.$

Proof. If we apply Hölder's inequality, we have

$$\begin{aligned}|F|\leq& \frac{(b-a)^{\alpha+1}}{2(\alpha+1)B(\alpha)\Gamma(\alpha)}\left(\int_0^1|m^{\alpha}(t)|^p\,dt\right)^{\frac{1}{p}}\\ &\times\left[\left(\int_0^1|f''(ta+(1-t)b)|^q\,dt\right)^{\frac{1}{q}}+\left(\int_0^1|f''(tb+(1-t)b)|^q\,dt\right)^{\frac{1}{q}}\right].\end{aligned}$$

Considering the variation of the HH-inequality for s-concave functions, we can state the findings below because $|f''|^q$ is s-concave on $[a,b]$:

$$\int_0^1|f''(ta+(1-t)b)|^q\,dt\leq 2^{\frac{1}{s}-1}\left|f''\left(\frac{a+b}{2}\right)\right|^q,$$
$$\int_0^1|f''(tb+(1-t)a)|^q\,dt\leq 2^{\frac{1}{s}-1}\left|f''\left(\frac{a+b}{2}\right)\right|^q.$$

We finish the proof by applying these findings to the aforementioned disparity.

REFERENCES

[1] Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, New York (2006).
[2] Set, E.: New inequalities of Ostrowski type for mappings whose derivatives are s-convex in the second sense via fractional integrals. Comput. Math. Appl. 63(7), 1147-1154 (2012).
[3] Set, E., Akdemir, A.O., Özata, F.: Grüss type inequalities for fractional integral operator involving the extended generalized Mittag-Leffler function. Appl. Comput. Math. 19(3), 402-414 (2020).
[4] Set, E., Akdemir, A.O., Özdemir, M.E.: Simpson type integral inequalities for convex functions via Riemann-Liouville integrals. Filomat 31(14), 4415-4420 (2017).
[5] Atangana, A., Baleanu, D.: New fractional derivatives with non-local and non-singular kernel: theory and application to heat transfer model. Therm. Sci. 20(2), 763-769 (2016).

[6] Hudzik, H. and Maligranda, L.: Some remarks ons-convex functions. Aequationes mathematicae 48.1 (1994): 100-111.
[7] Alomari, M., Darus, M. and Dragomir, S.S., New inequalities of Simpson's type for $s-$ convex functions with applications, *RGMIA Res. Rep. Coll.*, 12 (4) (2009), Article 9.
[8] Eken, Z., Fejer inequality fro s-convex functions in the fourth sense, Journal of Mathematical Inequalities, 2022, 16.1.
[9] Caputo, M. and Fabrizio, M.: A new definition of fractional derivative without singular kernel, *Progress in Fractional Differentiation and Applications*, vol. 1, no. 2, pp. 73-85, 2015.
[10] Abdeljawad, T. and Balenau, D.: On fractional derivatives with exponential kernel and their discrete versions, *Reports on Mathematical Physics*, vol. 80, no. 1, pp. 11-27, 2017.
[11] Abdeljawad, T. and Balenau, D.: Integration by parts and its applications of anew nonlocal fractional derivative with Mittag-Leffler nonsingular kernel," *The Journal of Nonlinear Sciences and Applications*, vol. 10, no. 3, pp. 1098-1107, 2017.
[12] Pecaric, J., Proschan, F., Tong, Y.L.: Convex Functions, Partial Orderings and Statistical Applications. Academic Press, San Diego (1992).
[13] Orlicz, W.: A note on modular spaces I. Bull. Acad. Pol. Sci., Sir. Sci. Math. Astron. Phys. 9, 157-162 (1961).
[14] Dragomir, S.S., Fitzpatrick, S.: The Hadamard inequalities for s-convex functions in the second sense. Demonstr. Math. 32(4), 687-696 (1999).
[15] Kavurmaci, H., Avci, M., Özdemir, M.E.: New inequalities of Hermite-Hadamard type for convex functions with applications. J. Inequal. Appl. 2011, 86 (2011)
[16] Özdemir, M.E., Ekinci, A., Akdemir, A.O.: Some new integral inequalities for functions whose derivatives of absolute values are convex and concave. TWMS J. Pure Appl. Math. 10(2), 212-224 (2019)
[17] Ozdemir, M.E., Latif, M.A., Akdemir, A.O.: On some Hadamard-type inequalities for product of two h-convex functions on the co-ordinates. Turk. J. Sci. 1(1), 41-58 (2016)
[18] Hussain, S., Bhatti, M.I., Iqbal, M.: Hadamard-type inequalities for s-convex functions I. Punjab Univ. J. Math. 41, 51-60 (2009)
[19] Dragomir, S.S., Pearce, C.E.M.: Selected Topics on Hermite-Hadamard Inequalities and Applications, RGMIA Monographs. Victoria University, Australia (2000)
[20] Guessab, A., Schmeisser, G.: Two Korovkin-type theorems in multivariate approximation. Banach J. Math. Anal. 2(2), 121-128 (2008)
[21] Kavurmacı Önalan, H., et al. On new general versions of Hermite-Hadamard type integral inequalities via fractional integral operators with Mittag-Leffler kernel. Journal of Inequalities and Applications 2021.1 (2021): 1-16.
[22] Kemali, S., Hermite-Hadamard Type Inequality for s-Convex Functions in the Fourth Sense. Turkish Journal of Mathematics and Computer Science 13.2 (2021): 287-293.

Some New Inequalities for Exponentially P- Functions on the Coordinates

Ahmet Ocak AKDEMİR[1], Sinan ASLAN[1] and Alper EKİNCİ[2]

[1]*Department of Mathematics, Faculty of Science and Arts,*
Ağrı İbrahim Çeçen University, Ağrı-TURKEY
[2]*Bandırma Vocational High School,*
Bandırma Onyedi Eylül University, Balıkesir-TURKEY

aocakakdemir@gmail.com
sinanaslan0407@gmail.com
alperekinci@hotmail.com

ABSTRACT

In this note, we defined a new class that is called exponentially P-functions which has a potential to produce novel estimations of Hadamard-type on the co-ordinates. Then, we have established some new Hermite-Hadamard type integral inequalities via exponentially P-functions on the coordinates.

INTRODUCTION

We will start by expressing an important inequality proved for convex functions. This inequality is presented on the basis of averages and give bounds for the mean value of a convex function.
Assume that $f: I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex mapping defined on the interval I of $\mathbb{R}$ where $a<b$. The following statement;

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

holds and known as Hermite-Hadamard inequality. Both inequalities hold in the reversed direction if f is concave.
In [1], Dragomir mentions an expansion of the concept of convex function, which is used in many inequalities in the field of inequality theory and has applications in different fields of mathematics, especially convex programming.

Definition 1 *Let us consider the bidimensional interval* $\Delta=[a,b]\times c,d]$ *in* $\mathbb{R}^2$ *with* $a<b$, $c<d$. *A function* $f:\Delta\to\mathbb{R}$ *will be called convex on the co-ordinates if the partial mappings* $f_y:[a,b]\to\mathbb{R}$, $f_y(u)=f(u,y)$ *and* $f_x:[c,d]\to\mathbb{R}$, $f_x(v)=f(x,v)$ *are convex where defined for all* $y\in[c,d]$ *and* $x\in[a,b]$ *Recall that the mapping* $f:\Delta\to\mathbb{R}$ *is convex on* Δ *if the following inequality holds,*

$$f(\lambda x+(1-\lambda)z,\lambda y+(1-\lambda)w)\leq \lambda f(x,y)+(1-\lambda)f(z,w)$$

for all $(x,y),(z,w)\in\Delta$ and $\lambda\in 0,1]$.

Expressing convex functions in coordinates brought up the question that it is possible for Hermite-Hadamard inequality to expand into coordinates. The answer to this motivating question has been found in Dragomir's paper (see [1]) and has taken its place in the literature as the expansion of Hermite-Hadamard inequality to a rectangle from the plane $\mathbb{R}^2$. stated below.

Theorem 1 *Suppose that* $f:\Delta=[a,b]\times c,d]\to\mathbb{R}$ *is convex on the co-ordinates on* Δ. *Then one has the inequalities;*

$$\begin{aligned}
&f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
&\leq\frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right]\\
&\leq\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy\\
&\leq\frac{1}{4}\left[\frac{1}{(b-a)}\int_a^b f(x,c)dx+\frac{1}{(b-a)}\int_a^b f(x,d)dx\right.\\
&\left.+\frac{1}{(d-c)}\int_c^d f(a,y)dy+\frac{1}{(d-c)}\int_c^d f(b,y)dy\right]\\
&\leq\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.
\end{aligned}\tag{1.1}$$

The above inequalities are sharp.

Numerous variants of this inequality were obtained for convexity and other types of convex functions in coordinates (See the papers [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 16]).

In [12], Sarıkaya *et al.* proved some Hadamard-type inequalities for co-ordinated convex functions as followings:

Theorem 2 *Let* $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ *be a partially differentiable mapping on* $\Delta:=[a,b]\times c,d]$ *in* $\mathbb{R}^2$ *with* $a<b$ *and* $c<d$. *If* $\left|\frac{\partial^2 f}{\partial t\partial s}\right|$ *is a convex function on the co-ordinates on* Δ, *then one has the inequalities:*

$$\begin{aligned}
&|J|\leq\frac{(b-a)(d-c)}{16}\\
&\times\frac{\left|\frac{\partial^2 f}{\partial t\partial s}\right|(a,c)+\left|\frac{\partial^2 f}{\partial t\partial s}\right|(a,d)+\left|\frac{\partial^2 f}{\partial t\partial s}\right|(b,c)+\left|\frac{\partial^2 f}{\partial t\partial s}\right|(b,d)}{4}
\end{aligned}\tag{1.2}$$

where

$$J = \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy - A$$

and

$$A = \frac{1}{2}\left[\frac{1}{(b-a)}\int_a^b [f(x,c)+f(x,d)]dx + \frac{1}{(d-c)}\int_c^d [f(a,y)dy+f(b,y)]dy\right].$$

Theorem 3 *Let* $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ *be a partially differentiable mapping on* $\Delta:=[a,b]\times c,d]$ *in* $\mathbb{R}^2$ *with* $a<b$ *and* $c<d$. *If* $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$, $q>1$, *is a convex function on the co-ordinates on* Δ, *then one has the inequalities:*

$$|J| \le \frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}} \times \left[\frac{\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q(a,c)+\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q(a,d)+\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q(b,c)+\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q(b,d)}{4}\right]^{\frac{1}{q}} \tag{1.3}$$

where A, J are as in Theorem 2 and $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 4 *Let* $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ *be a partially differentiable mapping on* $\Delta:=[a,b]\times c,d]$ *in* $\mathbb{R}^2$ *with* $a<b$ *and* $c<d$. *If* $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$, $q\ge 1$, *is a convex function on the co-ordinates on* Δ, *then one has the inequalities:*

$$|J| \le \frac{(b-a)(d-c)}{16} \times \left[\frac{\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q(a,c)+\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q(a,d)+\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q(b,c)+\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q(b,d)}{4}\right]^{\frac{1}{q}} \tag{1.4}$$

where A, J are as in Theorem 2.

In [17], Sarıkaya et al. have proved a new integral identity and several new inequalities as followings;

Lemma 1 *Let* $f:\Delta\subset R^2\to R$ *be a partial differentiable mapping on* $\Delta:=[a,b]\times[c,d]$ *in* R^2 *with* $a<b$ *and* $c<d$. *If* $\frac{\partial^2 f}{\partial t\partial s}\in L(\Delta)$, *then the following equality:*

$$\begin{aligned}&\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy\\&-\frac{1}{2}\left[\frac{1}{b-a}\int_a^b[f(x,c)+f(x,d)]dx+\frac{1}{d-c}\int_c^d[f(a,y)+f(b,y)]dy\right]\\&=\frac{(b-a)(d-c)}{4}\int_0^1\int_0^1(1-2t)(1-2s)\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b,sc+(1-s)d)dtds.\end{aligned}$$

Theorem 5 *Let* $f:\Delta=[a,b]\times[c,d]\to\mathrm{R}$ *be a partially differentiable mapping on* $\Delta=[a,b]\times[c,d]$. *If* $\left|\frac{\partial^2 f}{\partial t\partial s}\right|$ *is a convex function on the co-ordinates on* Δ, *then the following inequality holds;*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.\\&-\frac{1}{(d-c)}\int_c^d f\left(\frac{a+b}{2},y\right)dy-\frac{1}{(b-a)}\int_a^b f\left(x,\frac{c+d}{2}\right)dx\\&\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx\right|\\&\le\frac{(b-a)(d-c)}{64}\\&\times\left[\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\right].\end{aligned}\tag{1.5}$$

Theorem 6 *Let* $f:\Delta=[a,b]\times[c,d]\to\mathrm{R}$ *be a partially differentiable mapping on* $\Delta=[a,b]\times[c,d]$. *If* $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$, $q>1$, *is a convex function on the co-ordinates on* Δ, *then the following inequality holds;*

$$|C|\le\frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}}\tag{1.6}$$

$$\times\left[\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q}{4}\right]^{\frac{1}{q}}.$$

where

$$C = f\left(\frac{a+b}{2},\frac{c+d}{2}\right) -\frac{1}{(d-c)}\int_c^d f\left(\frac{a+b}{2},y\right)dy-\frac{1}{(b-a)}\int_a^b f\left(x,\frac{c+d}{2}\right)dx +\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx.$$

Theorem 7 *Let* $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ *be a partially differentiable mapping on* $\Delta=[a,b]\times[c,d]$*. If* $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$, $q\geq 1$, *is a convex function on the co-ordinates on* Δ, *then the following inequality holds;*

$$|C|\leq\frac{(b-a)(d-c)}{16} \tag{1.7}$$

$$\times\left[\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q}{4}\right]^{\frac{1}{q}}.$$

where

$$C = f\left(\frac{a+b}{2},\frac{c+d}{2}\right) -\frac{1}{(d-c)}\int_c^d f\left(\frac{a+b}{2},y\right)dy-\frac{1}{(b-a)}\int_a^b f\left(x,\frac{c+d}{2}\right)dx +\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx.$$

The concept of exponentially convex function in coordinates and the associated results are presented in the following form.

Definition 2 *(See [18]) Let us consider the interval such as* $\Delta=[\varepsilon_1,\varepsilon_2]\times\varepsilon_3,\varepsilon_4]$ *in* $\mathbb{R}^2$ *with* $\varepsilon_1<\varepsilon_2$, $\varepsilon_3<\varepsilon_4$. *The function* $\Psi:\Delta\to\mathbb{R}$ *is exponentially convex on* Δ *if*

$$\Psi((1-\zeta)u_1+\zeta u_3,(1-\zeta)u_2+\zeta u_4)\leq(1-\zeta)\frac{\Psi(u_1,u_2)}{e^{\alpha(u_1+u_2)}}+\zeta\frac{\Psi(u_3,u_4)}{e^{\alpha(u_3+u_4)}}$$

for all $(u_1,u_2),(u_3,u_4)\in\Delta,\ \alpha\in\mathrm{R}$ and $\zeta\in 0,1]$.

An equivalent definition of the exponentially convex function definition in coordinates can be done as follows:

Definition 3 *(See [18]) The mapping* $\Psi:\Delta\to\mathrm{R}$ *is exponentially convex function on the co-ordinates on* Δ*, if*

$$\Psi(\zeta\varepsilon_1+(1-\zeta)\varepsilon_2,\xi\varepsilon_3+(1-\xi)\varepsilon_4)$$

$$\leq\zeta\xi\frac{\Psi(\varepsilon_1,\varepsilon_3)}{e^{\alpha(\varepsilon_1+\varepsilon_3)}}+\zeta(1-\xi)\frac{\Psi(\varepsilon_1,\varepsilon_4)}{e^{\alpha(\varepsilon_1+\varepsilon_4)}}+(1-\zeta)\xi\frac{\Psi(\varepsilon_2,\varepsilon_3)}{e^{\alpha(\varepsilon_2+\varepsilon_3)}}+(1-\zeta)(1-\xi)\frac{\Psi(\varepsilon_2,\varepsilon_4)}{e^{\alpha(\varepsilon_2+\varepsilon_4)}}$$

for all $(\varepsilon_1,\varepsilon_3),(\varepsilon_1,\varepsilon_4),(\varepsilon_2,\varepsilon_3),(\varepsilon_2,\varepsilon_4)\in\Delta,\alpha\in\mathrm{R}$ and $\zeta,\xi\in[0,1]$.

EXPONENTIALLY $P-$FUNCTIONS ON THE CO-ORDINATES

Definition 4 *Let us consider the bidimensional interval* $\Delta=[a,b]\times c,d]$ *in* R^2 *with* $a<b$ *and* $c<d$. *The mapping* $f:\Delta\to R$ *is exponential* $P-$*function on the co-ordinates on* Δ*, if the following inequality holds*

$$f(tx+(1-t)z,ty+(1-t)w)\leq\frac{f(x,y)}{e^{\alpha(x+y)}}+\frac{f(z,w)}{e^{\alpha(z+w)}}$$

for all $(x,y),(z,w)\in\Delta,\alpha\in R$ and $t\in[0,1]$.

An equivalent definition of the exponential $P-$convex function definition in co-ordinates can be done as follows:

Definition 5 *The mapping* $f:\Delta\to R$ *is exponential* $P-$*convex on the co-ordinates on* Δ*, if the following inequality holds,*

$$f(ta+(1-t)b,sc+(1-s)d)\leq\frac{f(a,c)}{e^{\alpha(a+c)}}+\frac{f(a,d)}{e^{\alpha(a+d)}}+\frac{f(b,c)}{e^{\alpha(b+c)}}+\frac{f(b,d)}{e^{\alpha(b+d)}}.$$

for all $(a,c),(a,d),(b,c),(b,d)\in\Delta,\alpha\in R$ and $t,s\in[0,1]$

Lemma 2 *A function* $f:\Delta\to R$ *will be called exponential* $P-$ *function on the co-ordinates on* Δ*, if the partial mappings* $f_y:[a,b]\to R$ *,* $f_y(u)=e^{\alpha y}f(u,y)$ *and* $f_x:[c,d]\to R$ *,* $f_x(v)=e^{\alpha x}f(x,v)$ *are exponential* $P-$*function on the co-ordinates on* Δ*, where defined for all* $y\in[c,d]$ *and* $x\in[a,b]$.

Proof. From the definition of partial mapping f_x, we can write

$$
\begin{aligned}
& f_x(tv_1+(1-t)v_2)=e^{\alpha x} f(x,tv_1+(1-t)v_2) \\
& =e^{\alpha x} f(tx+(1-t)x,tv_1+(1-t)v_2) \\
& \leq e^{\alpha x}\left[\frac{f(x,v_1)}{e^{\alpha(x+v_1)}}+\frac{f(x,v_2)}{e^{\alpha(x+v_2)}}\right] \\
& =\frac{f(x,v_1)}{e^{\alpha v_1}}+\frac{f(x,v_2)}{e^{\alpha v_2}} \\
& =\frac{f_x(v_1)}{e^{\alpha v_1}}+\frac{f_x(v_2)}{e^{\alpha v_2}}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
& f_y(tu_1+(1-t)u_2)=e^{\alpha y} f(tu_1+(1-t)u_2,y) \\
& =e^{\alpha y} f(tu_1+(1-t)u_2,ty+(1-t)y) \\
& \leq e^{\alpha y}\left[\frac{f(u_1,y)}{e^{\alpha(u_1+y)}}+\frac{f(u_2,y)}{e^{\alpha(u_2+y)}}\right] \\
& =\frac{f(u_1,y)}{e^{\alpha u_1}}+\frac{f(u_2,y)}{e^{\alpha u_2}} \\
& =\frac{f_y(u_1)}{e^{\alpha u_1}}+\frac{f_y(u_2)}{e^{\alpha u_2}}.
\end{aligned}
$$

Proof is completed.

Theorem 8 *Let $f:\Delta=[a,b]\times c,d]\to R$ be partial differentiable mapping on $\Delta=[a,b]\times c,d]$ and $f\in L(\Delta)$, $\alpha\in R$. If f is exponential $P-$function on the co-ordinates on Δ, then the following inequality holds;*

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy\leq\frac{f(a,c)}{e^{\alpha(a+c)}}+\frac{f(a,d)}{e^{\alpha(a+d)}}+\frac{f(b,c)}{e^{\alpha(b+c)}}+\frac{f(b,d)}{e^{\alpha(b+d)}}.$$

Proof. By the definition of the exponential $P-$function on the co-ordinates on Δ, we can write

$$f\left(ta+(1-t)b,sc+(1-s)d\right)\leq\frac{f(a,c)}{e^{\alpha(a+c)}}+\frac{f(a,d)}{e^{\alpha(a+d)}}+\frac{f(b,c)}{e^{\alpha(b+c)}}+\frac{f(b,d)}{e^{\alpha(b+d)}}.$$

By integrating both sides of the above inequality with respect to t,s on $[0,1]^2$, we have

$$\int_0^1\int_0^1 f\left(ta+(1-t)b,sc+(1-s)d\right)dtds\leq\int_0^1\int_0^1\frac{f(a,c)}{e^{\alpha(a+c)}}+\frac{f(a,d)}{e^{\alpha(a+d)}}+\frac{f(b,c)}{e^{\alpha(b+c)}}+\frac{f(b,d)}{e^{\alpha(b+d)}}dtds$$

By computing the above integrals, we obtain the desired result.

Theorem 9 *Let* $f:\Delta=[a,b]\times c,d]\to R$ *be partial differentiable mapping on* $\Delta=[a,b]\times c,d]$ *and* $f\in L(\Delta)$, $\alpha\in R$. *If* $|f|$ *is exponential* $P-$*function on the co-ordinates on* Δ*,then the following inequality holds;*

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy\right|\leq\left|\frac{f(a,c)}{e^{\alpha(a+c)}}\right|+\left|\frac{f(a,d)}{e^{\alpha(a+d)}}\right|+\left|\frac{f(b,c)}{e^{\alpha(b+c)}}\right|+\left|\frac{f(b,d)}{e^{\alpha(b+d)}}\right|.$$

Proof. By the definition of the exponential $P-$function on the co-ordinates on Δ, we can write

$$\begin{aligned}&f\big(ta+(1-t)b,sc+(1-s)d\big)\\&\leq\frac{f(a,c)}{e^{\alpha(a+c)}}+\frac{f(a,d)}{e^{\alpha(a+d)}}+\frac{f(b,c)}{e^{\alpha(b+c)}}+\frac{f(b,d)}{e^{\alpha(b+d)}}\end{aligned}$$

The absolute value property is used in integral and by integrating both sides of the above inequality with respect to t,s on $[0,1]^2$, we can write

$$\begin{aligned}&\left|\int_0^1\int_0^1 f\big(ta+(1-t)b,sc+(1-s)d\big)dtds\right|\\&\leq\int_0^1\int_0^1\left|\frac{f(a,c)}{e^{\alpha(a+c)}}+\frac{f(a,d)}{e^{\alpha(a+d)}}+\frac{f(b,c)}{e^{\alpha(b+c)}}+\frac{f(b,d)}{e^{\alpha(b+d)}}\right|dtds.\end{aligned}$$

By the triangle inequality for integrals

$$\begin{aligned}&\left|\int_0^1\int_0^1 f\big(ta+(1-t)b,sc+(1-s)d\big)dtds\right|\\&\leq\int_0^1\int_0^1\left|\frac{f(a,c)}{e^{\alpha(a+c)}}\right|dtds+\int_0^1\int_0^1\left|\frac{f(a,d)}{e^{\alpha(a+d)}}\right|dtds\\&+\int_0^1\int_0^1\left|\frac{f(b,c)}{e^{\alpha(b+c)}}\right|dtds+\int_0^1\int_0^1\left|\frac{f(b,d)}{e^{\alpha(b+d)}}\right|dtds.\end{aligned}$$

If we apply the Hölder's inequality to the right-hand side of the inequality, we get

$$\begin{aligned}&\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy\right|\\&\leq\left(\int_0^1\int_0^1 dtds\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left|\frac{f(a,c)}{e^{\alpha(a+c)}}\right|^q dtds\right)^{\frac{1}{q}}\end{aligned}$$

$$\left(\int_0^1\int_0^1 dtds\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left|\frac{f(a,d)}{e^{\alpha(a+d)}}\right|^q dtds\right)^{\frac{1}{q}}$$

$$\left(\int_0^1\int_0^1 dtds\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left|\frac{f(b,c)}{e^{\alpha(b+c)}}\right|^q dtds\right)^{\frac{1}{q}}$$

$$\left(\int_0^1\int_0^1 dtds\right)^{\frac{1}{p}}\left(\int_0^1\int_0^1\left|\frac{f(b,d)}{e^{\alpha(b+d)}}\right|^q dtds\right)^{\frac{1}{q}}$$

By computing the above integrals, we obtain the desired result.

Theorem 10 *Let* $f:\Delta=[a,b]\times c,d]\to R$ *be partial differentiable mapping on* $\Delta=[a,b]\times c,d]$ *and* $f\in L(\Delta)$, $\alpha\in R$. *If* $|f|$ *is exponential* $P-$*function on the co-ordinates on* Δ, $p,q>1,\frac{1}{p}+\frac{1}{q}=1,$ *then the following inequality holds;*

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy\right|$$
$$\leq\frac{4}{p}+\frac{1}{q}\left(\left|\frac{f(a,c)}{e^{\alpha(a+c)}}\right|^q+\left|\frac{f(a,d)}{e^{\alpha(a+d)}}\right|^q+\left|\frac{f(b,c)}{e^{\alpha(b+c)}}\right|^q+\left|\frac{f(b,d)}{e^{\alpha(b+d)}}\right|^q\right).$$

Proof. By the definition of the exponential $P-$function on the co-ordinates on Δ, we can write

$$f\left(ta+(1-t)b,sc+(1-s)d\right)$$
$$\leq\frac{f(a,c)}{e^{\alpha(a+c)}}+\frac{f(a,d)}{e^{\alpha(a+d)}}+\frac{f(b,c)}{e^{\alpha(b+c)}}+\frac{f(b,d)}{e^{\alpha(b+d)}}.$$

If the absolute value property is used in integral and by integrating both sides of the above inequality with respect to t,s on $[0,1]^2$, we can write

$$\left|\int_0^1\int_0^1 f\left(ta+(1-t)b,sc+(1-s)d\right)dtds\right|$$
$$\leq\int_0^1\int_0^1\left|\frac{f(a,c)}{e^{\alpha(a+c)}}+\frac{f(a,d)}{e^{\alpha(a+d)}}+\frac{f(b,c)}{e^{\alpha(b+c)}}+\frac{f(b,d)}{e^{\alpha(b+d)}}\right|dtds.$$

By the triangle inequality for integrals

$$\left|\int_0^1\int_0^1 f\left(ta+(1-t)b,sc+(1-s)d\right)dtds\right|$$

$$\leq \int_0^1\int_0^1\left|\frac{f(a,c)}{e^{\alpha(a+c)}}\right|dtds+\int_0^1\int_0^1\left|\frac{f(a,d)}{e^{\alpha(a+d)}}\right|dtds$$
$$+\int_0^1\int_0^1\left|\frac{f(b,c)}{e^{\alpha(b+c)}}\right|dtds+\int_0^1\int_0^1\left|\frac{f(b,d)}{e^{\alpha(b+d)}}\right|dtds.$$

If we apply the Young's inequality to the right-hand side of the inequality, we get

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy\right|$$
$$\leq \frac{1}{p}\left(\int_0^1\int_0^1 dtds\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left|\frac{f(a,c)}{e^{\alpha(a+c)}}\right|^q dtds\right)$$
$$+\frac{1}{p}\left(\int_0^1\int_0^1 dtds\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left|\frac{f(a,d)}{e^{\alpha(a+d)}}\right|^q dtds\right)$$
$$+\frac{1}{p}\left(\int_0^1\int_0^1 dtds\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left|\frac{f(b,c)}{e^{\alpha(b+c)}}\right|^q dtds\right)$$
$$+\frac{1}{p}\left(\int_0^1\int_0^1 dtds\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left|\frac{f(b,d)}{e^{\alpha(b+d)}}\right|^q dtds\right).$$

By computing the above integrals, we obtain the desired result.

Proposition 1 *If* $f,g:\Delta\to R$ *are two exponential* $P-$*function on the co-ordinates on* Δ, *then* $f+g$ *are exponential* $P-$*convex functions on the co-ordinates on* Δ.

Proof. By the definition of the exponential $P-$function on the co-ordinates on Δ, we can write

$$f(ta+(1-t)b,sc+(1-s)d)+g(ta+(1-t)b,sc+(1-s)d)$$
$$\leq\left(\frac{f(a,c)}{e^{\alpha(a+c)}}+\frac{g(a,c)}{e^{\alpha(a+c)}}\right)+\left(\frac{f(a,d)}{e^{\alpha(a+d)}}+\frac{g(a,d)}{e^{\alpha(a+d)}}\right)$$
$$\left(\frac{f(b,c)}{e^{\alpha(b+c)}}+\frac{g(b,c)}{e^{\alpha(b+c)}}\right)+\left(\frac{f(b,d)}{e^{\alpha(b+d)}}+\frac{g(b,d)}{e^{\alpha(b+d)}}\right).$$

Namely,

$$(f+g)(ta+(1-t)b,sc+(1-s)d)$$
$$\leq\frac{(f+g)(a,c)}{e^{\alpha(a+c)}}+\frac{(f+g)(a,d)}{e^{\alpha(a+d)}}+\frac{(f+g)(b,c)}{e^{\alpha(b+c)}}+\frac{(f+g)(b,d)}{e^{\alpha(b+d)}}.$$

Therefore (f+g) is exponential p-function on the co-ordinates on Δ.

Proposition 2 *If* $f:\Delta\to R$ *is exponential p-function on the co-ordinates on* Δ *and* $k\geq 0$ *then* kf *is exponential* $P-$*function on the co-ordinates on* Δ.

Proof. By the definition of the exponential $P-$function on the co-ordinates on Δ, we can write

$$f\left(ta+(1-t)b, sc+(1-s)d\right) \leq \frac{f(a,c)}{e^{\alpha(a+c)}}+\frac{f(a,d)}{e^{\alpha(a+d)}}+\frac{f(b,c)}{e^{\alpha(b+c)}}+\frac{f(b,d)}{e^{\alpha(b+d)}}.$$

If both sides are multiplied by k, we have,

$$(kf)(ta+(1-t)b, sc+(1-s)d) \leq \frac{(kf)(a,c)}{e^{\alpha(a+c)}}+\frac{(kf)(a,d)}{e^{\alpha(a+d)}}+\frac{(kf)(b,c)}{e^{\alpha(b+c)}}+\frac{(kf)(b,d)}{e^{\alpha(b+d)}}.$$

Therefore (kf) is exponential $P-$function on the co-ordinates on Δ .

RELATED RESULTS FOR EXPONENTIALLY P-FUNCTIONS ON THE CO-ORDINATES

Theorem 11 *Let $f:\Delta\subset R^2\to R$ be a partial differentiable mapping on $\Delta:=[a,b]\times[c,d]$ in R^2 with $a<b$ and $c<d$. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|$ is a exponentially P-function on the co-ordinates on Δ, then one has the inequality:*

$$|E|\leq\frac{(b-a)(d-c)}{16}\times\left(\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|}{e^{\alpha(a+c)}}+\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|}{e^{\alpha(a+d)}}+\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|}{e^{\alpha(b+c)}}+\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|}{e^{\alpha(b+d)}}\right).$$

where

$$E=\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy$$
$$-\frac{1}{2}\left[\frac{1}{b-a}\int_a^b[f(x,c)+f(x,d)]dx+\frac{1}{d-c}\int_c^d[f(a,y)+f(b,y)]dy\right].$$

Proof. From Lemma 2, we have

$$|E|\leq\frac{(b-a)(d-c)}{4}\times\int_0^1\int_0^1|(1-2t)(1-2s)|\left|\frac{\partial^2 f}{\partial t\partial s}\left(ta+(1-t)b, sc+(1-s)d\right)\right|dtds.$$

f is co-ordinated exponentially P-function on Δ , then one has:

$$|E| \le \frac{(b-a)(d-c)}{4} \times \int_0^1\int_0^1 |(1-2t)(1-2s)| \left(\frac{\left|\frac{\partial^2 f}{\partial t \partial s}(a,c)\right|}{e^{\alpha(a+c)}} + \frac{\left|\frac{\partial^2 f}{\partial t \partial s}(a,d)\right|}{e^{\alpha(a+d)}} + \frac{\left|\frac{\partial^2 f}{\partial t \partial s}(b,c)\right|}{e^{\alpha(b+c)}} + \frac{\left|\frac{\partial^2 f}{\partial t \partial s}(b,d)\right|}{e^{\alpha(b+d)}} \right) dtds.$$

By calculating the integral in above inequality, we have

$$|E| \le \frac{(b-a)(d-c)}{16} \times \left(\frac{\left|\frac{\partial^2 f}{\partial t \partial s}(a,c)\right|}{e^{\alpha(a+c)}} + \frac{\left|\frac{\partial^2 f}{\partial t \partial s}(a,d)\right|}{e^{\alpha(a+d)}} + \frac{\left|\frac{\partial^2 f}{\partial t \partial s}(b,c)\right|}{e^{\alpha(b+c)}} + \frac{\left|\frac{\partial^2 f}{\partial t \partial s}(b,d)\right|}{e^{\alpha(b+d)}} \right).$$

Theorem 12 *Let* $f : \Delta \subset R^2 \to R$ *be a partial differentiable mapping on* $\Delta := [a,b] \times [c,d]$ *in* R^2 *with* $a < b$ *and* $c < d$*. If* $\left|\frac{\partial^2 f}{\partial t \partial s}\right|^q$ *is a exponentially P-function on the co-ordinates on* Δ, $p, q > 1, \frac{1}{p} + \frac{1}{q} = 1$ *then one has the inequalities:*

$$|E| \le \frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}} \left(\frac{\left|\frac{\partial^2 f}{\partial t \partial s}(a,c)\right|^q}{e^{q\alpha(a+c)}} + \frac{\left|\frac{\partial^2 f}{\partial t \partial s}(a,d)\right|^q}{e^{q\alpha(a+d)}} + \frac{\left|\frac{\partial^2 f}{\partial t \partial s}(b,c)\right|^q}{e^{q\alpha(b+c)}} + \frac{\left|\frac{\partial^2 f}{\partial t \partial s}(b,d)\right|^q}{e^{q\alpha(b+d)}} \right)^{\frac{1}{q}}$$

where

$$E = \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4} + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\,dxdy$$
$$\frac{1}{2}\left[\frac{1}{b-a} \int_a^b [f(x,c) + f(x,d)]dx + \frac{1}{d-c} \int_c^d [f(a,y) + f(b,y)]dy \right] -.$$

Proof. From Lemma 2, we have

$$|E| \leq \frac{(b-a)(d-c)}{4}$$
$$\times \int_0^1 \int_0^1 |(1-2t)(1-2s)| \left| \frac{\partial^2 f}{\partial t \partial s}(ta+(1-t)b, sc+(1-s)d) \right| dtds.$$

If we apply the Hölder's inequality to the right-hand side of the inequality, we get

$$|E| \leq \frac{(b-a)(d-c)}{4}$$
$$\times \left(\int_0^1 \int_0^1 |(1-2t)(1-2s)|^p \, dtds \right)^{\frac{1}{p}} \left(\int_0^1 \int_0^1 \left| \frac{\partial^2 f}{\partial t \partial s}(ta+(1-t)b, sc+(1-s)d) \right|^q dtds \right)^{\frac{1}{q}}$$

f is co-ordinated exponentially P-function on Δ, then one has:

$$|E| \leq \frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}} \left(\int_0^1 \int_0^1 \frac{\left| \frac{\partial^2 f}{\partial t \partial s}(a,c) \right|^q}{e^{q\alpha(a+c)}} + \frac{\left| \frac{\partial^2 f}{\partial t \partial s}(a,d) \right|^q}{e^{q\alpha(a+d)}} \right.$$
$$\left. + \frac{\left| \frac{\partial^2 f}{\partial t \partial s}(b,c) \right|^q}{e^{q\alpha(b+c)}} + \frac{\left| \frac{\partial^2 f}{\partial t \partial s}(b,d) \right|^q}{e^{q\alpha(b+d)}} \, dtds \right)^{\frac{1}{q}}$$

hence, it follows that

$$|E| \leq \frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}} \left(\frac{\left| \frac{\partial^2 f}{\partial t \partial s}(a,c) \right|^q}{e^{q\alpha(a+c)}} + \frac{\left| \frac{\partial^2 f}{\partial t \partial s}(a,d) \right|^q}{e^{q\alpha(a+d)}} + \frac{\left| \frac{\partial^2 f}{\partial t \partial s}(b,c) \right|^q}{e^{q\alpha(b+c)}} + \frac{\left| \frac{\partial^2 f}{\partial t \partial s}(b,d) \right|^q}{e^{q\alpha(b+d)}} \right)^{\frac{1}{q}}$$

Theorem 13 *Let* $f : \Delta \subset R^2 \to R$ *be a partial differentiable mapping on* $\Delta := [a,b] \times [c,d]$ *in* R^2 *with* $a < b$ *and* $c < d$. *If* $\left| \frac{\partial^2 f}{\partial t \partial s} \right|^q$ *is a exponentially P-function on the co-ordinates on* Δ, $p, q > 1, \frac{1}{p} + \frac{1}{q} = 1$ *then one has the inequalities:*

$$|E|\leq\frac{(b-a)(d-c)}{4}\left(\frac{1}{p(p+1)^2}+\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q}{qe^{q\alpha(a+c)}}+\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q}{qe^{q\alpha(a+d)}}+\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q}{qe^{q\alpha(b+c)}}+\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q}{qe^{q\alpha(b+d)}}\right)$$

where

$$E=\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy$$
$$-\frac{1}{2}\left[\frac{1}{b-a}\int_a^b[f(x,c)+f(x,d)]dx+\frac{1}{d-c}\int_c^d[f(a,y)+f(b,y)]dy\right].$$

Proof. From Lemma 2, we have

$$|E|\leq\frac{(b-a)(d-c)}{4}$$
$$\times\int_0^1\int_0^1|(1-2t)(1-2s)|\left|\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b,sc+(1-s)d)\right|dtds.$$

If we apply the Young's inequality to the right-hand side of the inequality, we get

$$|E|\leq\frac{(b-a)(d-c)}{4}$$
$$\times\left(\frac{1}{p}\left(\int_0^1\int_0^1|(1-2t)(1-2s)|^p dtds\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b,sc+(1-s)d)\right|^q dtds\right)\right)$$

f is co-ordinated exponentially P-function on Δ, then one has:

$$|E|\leq\frac{(b-a)(d-c)}{4}\left(\frac{1}{p(p+1)^2}+\frac{1}{q}\left(\int_0^1\int_0^1\left|\frac{\frac{\partial^2 f}{\partial t\partial s}(a,c)}{e^{\alpha(a+c)}}\right|^q+\left|\frac{\frac{\partial^2 f}{\partial t\partial s}(a,d)}{e^{\alpha(a+d)}}\right|^q\right.\right.$$
$$\left.\left.+\left|\frac{\frac{\partial^2 f}{\partial t\partial s}(b,c)}{e^{\alpha(b+c)}}\right|^q+\left|\frac{\frac{\partial^2 f}{\partial t\partial s}(b,d)}{e^{\alpha(b+d)}}\right|^q dtds\right)\right)$$

hence, it follows that

$$|E| \leq \frac{(b-a)(d-c)}{4}\left(\frac{1}{p(p+1)^2}+\frac{\left|\frac{\partial^2 f}{\partial t \partial s}(a,c)\right|^q}{qe^{q\alpha(a+c)}}+\frac{\left|\frac{\partial^2 f}{\partial t \partial s}(a,d)\right|^q}{qe^{q\alpha(a+d)}}+\frac{\left|\frac{\partial^2 f}{\partial t \partial s}(b,c)\right|^q}{qe^{q\alpha(b+c)}}+\frac{\left|\frac{\partial^2 f}{\partial t \partial s}(b,d)\right|^q}{qe^{q\alpha(b+d)}}\right)$$

ACKNOWLEDGEMENTS

This study was supported by Ağrı İbrahim Çeçen University BAP Coordination Unit with the Project number FEF.22.004.

REFERENCES

[1] S.S. Dragomir, On Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane, Taiwanese Journal of Math., 5, 2001, 775-788.
[2] M.E. Özdemir, M.A. Latif, A.O. Akdemir, On some Hadamard-type inequalities for product of two $s-$convex functions on the co-ordinates, Journal of Inequalities and Applications , 2012, February-2012, DOI: 10.1186/1029-242X-2012-21.
[3] E. Set, M.Z. Sarikaya, A.O. Akdemir, A new general inequality for double integrals, American Institu of Phsyics (AIP) Conference Proceedings , 1470, 122-125 pp., October-2012, DOI: http://dx.doi.org/10.1063/1.4747655.
[4] M. Alomari and M. Darus, Hadamard-Type inequalities for $s-$convex functions, International Mathematical Forum, (3) 2008, no. 40, 1965 - 1975.
[5] M.E. Özdemir, A.O. Akdemir, On the Hadamard-type inequalities involving product of two convex functions on the co-ordinates, Tamkang Journal of Mathematics, vol. 46, no. 2, pp. 129-142, 2015.
[6] A.O. Akdemir, M.E. Özdemir, Some Hadamard-type inequalities for co-ordinated $P-$convex functions and Godunova-Levin functions, American Institu of Phsyics (AIP) Conference Proceedings , 1309, 7-15 pp., 2010.
[7] M. E. Özdemir, M.A. Latif, A.O. Akdemir, On some Hadamard-type inequalities for product of two $h-$convex functions on the co-ordinates, Turkish Journal of Science, 1 (1), 2016, 48-58.
[8] M.K. Bakula and J. Pecaric, On the Jensen's inequality for convex functions on the co-ordinates in a rectangle from the plane, Taiwanese Journal of Math., 5, 2006, 1271-1292.
[9] M.E. Özdemir, A.O. Akdemir, C. Yıldız, On co-ordinated Quasi-convex functions, Czechoslovak Mathematical Journal , 62-4, 889-900 pp., 2012.
[10] M. Alomari and M. Darus, The Hadamard's inequality for $s-$convex functions of $2-$variables, Int. Journal of Math. Analysis, 2(13), 2008, 629-638.
[11] M.E. Özdemir, E. Set and M.Z. Sarıkaya, Some new Hadamard's type inequalities for co-ordinated $m-$convex and $(\alpha,m)-$convex functions, Hacettepe J. of. Math. and St., 40, 219-229, (2011).
[12] M.Z. Sarıkaya, E. Set, M. Emin Özdemir and S.S. Dragomir, New some Hadamard's type inequalities for co-ordinated convex functions, Tamsui Oxford J. Math. Sci., 28, (2011).
[13] M.E. Özdemir, C. Yıldız, A.O. Akdemir, On some new Hadamard-type inequalities for co-ordinated Quasi-convex cunctions, Hacettepe Journal of Mathematics and Statistics, 41-5, 697-707 pp., 2012.

[14] S.S. Dragomir & C. Pearce, Selected topics on Hermite-Hadamard inequalities and applications. Victoria University: RGMIA Monographs, (17) 2000. [http://ajmaa.org/RGMIA/monographs/hermite hadamard.html].

[15] M.E. Özdemir, H. Kavurmacı, A.O. Akdemir, and M. Avcı, Inequalities for convex and $s-$convex functions on $\Delta=[a,b]\times[c,d]$, J. Inequal. Appl., (2012), 2012:20.

[16] M.E. Özdemir, C. Yıldız, A.O. Akdemir, On the co-ordinated convex functions, Applied Mathematics and Information Sciences , 8-3, 1085-1091 pp., 2014, DOI: 10.12785/amis/080318.

[17] M.Z. Sarıkaya, E. Set, M.E. Ozdemir, S.S. Dragomir, New Some Hadamard's Type Inequalities for Co-ordinated Convex Functions. Tamsui Oxford Journal of Information & Mathematical Sciences (TOJIMS), 28(2),

[18] S. Aslan and A.O. Akdemir, Exponentially convex functions on the co-ordinates and related integral inequalities, Proceedings of the 8th International Conference on Control and Optimization with Industrial Applications, Volume II, pp. 120-122, 24-26 August 2022. 137-152, 2012.

On Simpson's Type Inequalities for Quasi-Convex Functions via Atangana-Baleanu Integral Operators

M. Emin ÖZDEMİR[1], Merve Nur ÇAKALOĞLU[2], Ahmet Ocak AKDEMİR[3] and Abdullah ERGÜN[4]

[1] Department of Mathematics Education, Faculty of Education, Bursa Uludağ University, Bursa-TURKEY
[2]Department of Mathematics Education, Institute of Graduate Studies, Ağrı İbrahim Çeçen University, Ağrı-TURKEY
[3]Department of Mathematics, Faculty of Science and Letters, Ağrı İbrahim Çeçen University, Ağrı-TURKEY
[4]Sivas Cumhuriyet University, Vocational School of Sivas, 58140, Sivas-TURKEY

eminozdemir@uludag.edu.tr
aocakakdemir@gmail.com
merve.nur.cakaloglu@gmail.com
aergun@cumhuriyet.edu.tr

ABSTRACT

In the present note, several novel estimations of Simpson's type have been presented by using an integral identity that includes Atangana-Baleanu fractional integral operators for quasi-convex functions. We have used the basic definitions, some classical inequalities and elementary analysis methods.

INTRODUCTION

Suppose $f:[a,b]\to \mathbb{R}$ is a four times continuously differentiable mapping on (a,b) and $\left\|f^{(4)}\right\|_\infty = \sup\left|f^{(4)}(x)\right| < \infty$. The following inequality

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{1}{2880}\left\|f^{(4)}\right\|_\infty (b-a)^4$$

is well known in the literature as Simpson's inequality.
For some recent results related to Simpson's inequality see [1]-[5] and [7].
The function $f:[a,b]\to\mathbb{R}$, is said to be convex, if we have

$$f(tx+(1-t)y)\leq tf(x)+(1-t)f(y)$$

for all $x,y\in[a,b]$ and $t\in[0,1]$.

Convex functions play an important role in many branches of mathematics and the other sciences as engineering, economics and optimization theory. Several extensions, generalizations and refinements have been presented by researchers.

Definition 1. *([6]) A function $f:[0,b]\rightarrow(0,\infty)$ is said to be $m-$logarithmically convex if the inequality*

$$f\left(tx+m(1-t)y\right)\leq[f(x)]^{t}[f(y)]^{m(1-t)} \tag{1.1}$$

holds for all $x,y\in[0,b]$, $m\in(0,1]$, and $t\in 0,1]$.
Obviously, if we set $m=1$ in Definition 1, then f is just the ordinary logarithmically convex function on $[0,b]$.

Definition 2. *([6]) A function $f:[0,b]\rightarrow(0,\infty)$ is said to be $(\alpha,m)-$ logarithmically convex if*

$$f\left(tx+m(1-t)y\right)\leq[f(x)]^{t^{\alpha}}[f(y)]^{m\left(1-t^{\alpha}\right)} \tag{1.2}$$

holds for all $x,y\in 0,b]$, $(\alpha,m)\in(0,1]\times(0,1]$, and $t\in 0,1]$.

Clearly, when taking $\alpha=1$ in Definition 2, then f becomes the standard $m-$logarithmically convex function on $[0,b]$.

Definition 3. (Atangana-Baleanu Fractional Derivative) $f\in L^{1}(\phi,\omega)$, $\phi<\omega$, $\varepsilon\in[0,1]$, then the fractional derivative can be defined as

$$ {}_{\phi}^{AB}D_{\kappa}^{\varepsilon}f(\kappa)=\frac{B(\varepsilon)}{1-\varepsilon}\int_{\phi}^{\kappa}f'(k)E_{\varepsilon}\left[-\varepsilon\frac{(\kappa-k)^{\varepsilon}}{(1-\varepsilon)}\right]dk$$

$$ {}_{\omega}^{AB}D_{\kappa}^{\varepsilon}f(\kappa)=\frac{B(\varepsilon)}{1-\varepsilon}\int_{\kappa}^{\omega}f'(k)E_{\varepsilon}\left[-\varepsilon\frac{(k-\kappa)^{\varepsilon}}{(1-\varepsilon)}\right]dk$$

where $\mathrm{B}(\varepsilon)$ is normalization function and $\mathrm{B}(\varepsilon)>0$, $\mathrm{B}(0)=\mathrm{B}(1)=1$ (Abdeljawad & Baleanu, 2017).
Definition 4. (Atangana-Baleunu Fractional Integral Operator) $f\in L^{1}(\phi,\omega)$, $\phi<\omega$, $\varepsilon\in[0,1]$, the associated fractional integral operator can be given as:

$$ {}_{\phi}^{AB}I_{\kappa}^{\varepsilon}\{f(\kappa)\}=\frac{1-\varepsilon}{\mathrm{B}(\varepsilon)}f(\kappa)+\frac{\varepsilon}{\mathrm{B}(\varepsilon)\Gamma(\varepsilon)}\int_{\phi}^{\kappa}f(k)(\kappa-k)^{\varepsilon-1}dk$$

$$ {}_{\omega}^{AB}I_{\kappa}^{\varepsilon}\{f(\kappa)\}=\frac{1-\varepsilon}{\mathrm{B}(\varepsilon)}f(\kappa)+\frac{\varepsilon}{\mathrm{B}(\varepsilon)\Gamma(\varepsilon)}\int_{\kappa}^{\omega}f(k)(k-\kappa)^{\varepsilon-1}dk$$

where $\mathrm{B}(\varepsilon)$ is normalization function and $\mathrm{B}(\varepsilon)>0$, $\mathrm{B}(0)=\mathrm{B}(1)=1$ (Abdeljawad & Baleanu, 2017).

NEW RESULTS

Lemma 1. Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) with $a<b$ and $f'\in L[a,b]$. Then, we have the following identity for Atangana-Baleanu fractional integral operators:

$$\begin{aligned}
I &= \frac{(b-a)^{\xi}+2^{\xi+1}(1-\xi)\Gamma(\xi)}{3.2^{\xi}(b-a)\mathrm{B}(\xi)\Gamma(\xi)}\big(f(a)+f(b)\big)+\frac{(b-a)^{\xi}+2^{\xi-1}(1-\xi)\Gamma(\xi)}{3.2^{\xi-2}(b-a)\mathrm{B}(\xi)\Gamma(\xi)}f\left(\frac{a+b}{2}\right)\\
&\qquad -\frac{2}{3(b-a)}\left[{}_{\frac{a+b}{2}}^{AB}I^{\xi}f(b)+{}^{AB}I_{\frac{a+b}{2}}^{\xi}f(a)\right]\\
&\qquad -\frac{1}{3(b-a)}\left[{}^{AB}I_{b}^{\xi}f\left(\frac{a+b}{2}\right)+{}_{a}^{AB}I^{\xi}f\left(\frac{a+b}{2}\right)\right]\\
&= -\frac{(b-a)^{\xi}}{2^{\xi+1}\mathrm{B}(\xi)\Gamma(\xi)}\left[\int_0^1\frac{2(1-t)^{\xi}-t^{\xi}}{3}f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)dt\right.\\
&\qquad \left.+\int_0^1\frac{t^{\xi}-2(1-t)^{\xi}}{3}f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt\right]
\end{aligned}$$

where $\xi\in(0,1]$, $t\in[0,1]$, $B(\xi)$ is the normalization function.

Proof. Integration by parts, we have

$$I_1=\int_0^1\frac{2(1-t)^{\xi}-t^{\xi}}{3}f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)dt$$

$$\begin{aligned}
&= -\frac{4}{3(b-a)}f\left(\frac{a+b}{2}\right)+\frac{2^{\xi+2}\xi}{3(b-a)^{\xi+1}}\int_{\frac{a+b}{2}}^{b}(b-u)^{\xi-1}f(u)du-\frac{2}{3(b-a)}f(b)\\
&\qquad +\frac{2^{\xi+1}\xi}{3(b-a)^{\xi+1}}\int_{\frac{a+b}{2}}^{b}\left(u-\frac{a+b}{2}\right)^{\xi-1}f(u)du
\end{aligned}$$

and

$$I_2=\int_0^1\frac{t^{\xi}-2(1-t)^{\xi}}{3}f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt$$

$$\begin{aligned}
&= -\frac{2}{3(b-a)}f(a)+\frac{2^{\xi+1}\xi}{3(b-a)^{\xi+1}}\int_{a}^{\frac{a+b}{2}}\left(\frac{a+b}{2}-u\right)^{\xi-1}f(u)du-\frac{4}{3(b-a)}f\left(\frac{a+b}{2}\right)\\
&\qquad +\frac{2^{\xi+2}\xi}{3(b-a)^{\xi+1}}\int_{a}^{\frac{a+b}{2}}(u-a)^{\xi-1}f(u)du
\end{aligned}$$

By adding I_1 and I_2 and multiplying the both sides $-\frac{(b-a)^{\xi}}{2^{\xi+1}B(\xi)\Gamma(\xi)}$, we get desired results.

Theorem 1. Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) with $a<b$ and $f'\in L[a,b]$. If $|f'|$ is a convex function, we have the following inequality for Atangana-Baleanu fractional integral operators:

$$\begin{aligned}
&\frac{(b-a)^{\xi}+2^{\xi+1}(1-\xi)\Gamma(\xi)}{3.2^{\xi}(b-a)B(\xi)\Gamma(\xi)}(f(a)+f(b))+\frac{(b-a)^{\xi}+2^{\xi-1}(1-\xi)\Gamma(\xi)}{3.2^{\xi-2}(b-a)B(\xi)\Gamma(\xi)}f\left(\frac{a+b}{2}\right)\\
&\qquad-\frac{2}{3(b-a)}\left[{}^{AB}_{\frac{a+b}{2}}I^{\xi}f(b)+{}^{AB}I^{\xi}_{\frac{a+b}{2}}f(a)\right]\\
&\qquad-\frac{1}{3(b-a)}\left[{}^{AB}I^{\xi}_{b}f\left(\frac{a+b}{2}\right)+{}^{AB}_{a}I^{\xi}f\left(\frac{a+b}{2}\right)\right]\\
&\leq\frac{(b-a)^{\xi}}{2^{\xi+1}B(\xi)\Gamma(\xi)}\left[\frac{-4\left(1-\frac{2^{\frac{1}{\xi}}}{2^{\frac{1}{\xi}}+1}\right)^{\xi+1}-2\left(\frac{2^{\frac{1}{\xi}}}{2^{\frac{1}{\xi}}+1}\right)^{\xi+1}+3}{3(\xi+1)}\right](|f'(a)|+|f'(b)|)
\end{aligned}$$

where $\xi\in(0,1], B(\xi)$ is the normalization function.

Proof. From the integral identity given in Lemma 1 and by using the properties of modulus, we have

$$\begin{aligned}
|I|\leq\frac{(b-a)^{\xi}}{2^{\xi+1}B(\xi)\Gamma(\xi)}\Bigg(&\int_0^1\left|\frac{2(1-t)^{\xi}-t^{\xi}}{3}\right|\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|dt\\
&+\int_0^1\left|\frac{t^{\xi}-2(1-t)^{\xi}}{3}\right|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt\Bigg)
\end{aligned}$$

By using convexity of $|f'|$, we get

$$\begin{aligned}
|I|\leq\frac{(b-a)^{\xi}}{2^{\xi+1}B(\xi)\Gamma(\xi)}\Bigg(&\int_0^1\left|\frac{2(1-t)^{\xi}-t^{\xi}}{3}\right|\left(\frac{1+t}{2}|f'(b)|+\frac{1-t}{2}|f'(a)|\right)dt\\
&+\int_0^1\left|\frac{t^{\xi}-2(1-t)^{\xi}}{3}\right|\left(\frac{1+t}{2}|f'(a)|+\frac{1-t}{2}|f'(b)|\right)dt\Bigg)
\end{aligned}$$

$$
= \frac{(b-a)^{\xi}}{2^{\xi+1}\mathrm{B}(\xi)\Gamma(\xi)}\left(\int_0^{\frac{2^{\frac{1}{\xi}}}{2^{\frac{1}{\xi}}+1}}\left(\frac{2(1-t)^{\xi}-t^{\xi}}{3}\right)\left(\frac{1+t}{2}|f'(b)|+\frac{1-t}{2}|f'(a)|\right)dt\right.
$$

$$
+\int_{\frac{2^{\frac{1}{\xi}}}{2^{\frac{1}{\xi}}+1}}^{1}\left(\frac{t^{\xi}-2(1-t)^{\xi}}{3}\right)\left(\frac{1+t}{2}|f'(b)|+\frac{1-t}{2}|f'(a)|\right)dt
$$

$$
+\int_0^{\frac{2^{\frac{1}{\xi}}}{2^{\frac{1}{\xi}}+1}}\left(\frac{2(1-t)^{\xi}-t^{\xi}}{3}\right)\left(\frac{1+t}{2}|f'(a)|+\frac{1-t}{2}|f'(b)|\right)dt
$$

$$
\left.+\int_{\frac{2^{\frac{1}{\xi}}}{2^{\frac{1}{\xi}}+1}}^{1}\left(\frac{t^{\xi}-2(1-t)^{\xi}}{3}\right)\left(\frac{1+t}{2}|f'(a)|+\frac{1-t}{2}|f'(b)|\right)dt\right)
$$

By computing the above integral, we obtain

$$
|I| \le \frac{(b-a)^{\xi}}{2^{\xi+1}\mathrm{B}(\xi)\Gamma(\xi)}\left[\frac{-4\left(1-\frac{2^{\frac{1}{\xi}}}{2^{\frac{1}{\xi}}+1}\right)^{\xi+1}-2\left(\frac{2^{\frac{1}{\xi}}}{2^{\frac{1}{\xi}}+1}\right)^{\xi+1}+3}{3(\xi+1)}\right](|f'(a)|+|f'(b)|)
$$

Theorem 2. Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) with $a<b$ and $f'\in L[a,b]$. If $|f'|^{q}$ is a convex function, we have the following inequality for Atangana-Baleanu fractional integral operators:

$$|I| \leq \frac{(b-a)^{\xi}}{2^{\xi+1}\mathrm{B}(\xi)\Gamma(\xi)}\left(p.\frac{3-2\left(\frac{2^{\frac{1}{\xi}}}{2^{\frac{1}{\xi}}+1}\right)^{\xi+1}-4\left(1-\frac{2^{\frac{1}{\xi}}}{2^{\frac{1}{\xi}}+1}\right)^{\xi+1}}{3(\xi+1)}\right)^{\frac{1}{p}}\left[\left(\frac{3}{4}|f'(b)|^{q}\right.\right.$$
$$\left.+\frac{1}{4}|f'(a)|^{q}\right)^{\frac{1}{q}}+\left(\frac{3}{4}|f'(a)|^{q}+\frac{1}{4}|f'(b)|^{q}\right)^{\frac{1}{q}}\Bigg]$$

where $p^{-1}+q^{-1}=1$, $\xi\in(0,1]$, $q>1$ and $\mathrm{B}(\xi)$ is the normalization function.

Proof. By using the identity that is given Lemma 1, we have

$$|I| \leq \frac{(b-a)^{\xi}}{2^{\xi+1}\mathrm{B}(\xi)\Gamma(\xi)}\left(\int_0^1\left|\frac{2(1-t)^{\xi}-t^{\xi}}{3}\right|\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|dt\right.$$
$$\left.+\int_0^1\left|\frac{t^{\xi}-2(1-t)^{\xi}}{3}\right|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt\right)$$

By applying Hölder inequality, we have

$$|I| \leq \frac{(b-a)^{\xi}}{2^{\xi+1}\mathrm{B}(\xi)\Gamma(\xi)}\left[\left(\int_0^1\left|\frac{2(1-t)^{\xi}-t^{\xi}}{3}\right|^{p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^{q}dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\left|\frac{t^{\xi}-2(1-t)^{\xi}}{3}\right|^{p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^{q}dt\right)^{\frac{1}{q}}\right]$$

By using convexity of $|f'|^{q}$, we obtain

$$|I| \leq \frac{(b-a)^{\xi}}{2^{\xi+1}\mathrm{B}(\xi)\Gamma(\xi)}\left[\left(\int_0^1\left|\frac{2(1-t)^{\xi}-t^{\xi}}{3}\right|^{p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left(\frac{1+t}{2}|f'(b)|^{q}+\frac{1-t}{2}|f'(a)|^{q}\right)dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\left|\frac{t^{\xi}-2(1-t)^{\xi}}{3}\right|^{p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left(\frac{1+t}{2}|f'(a)|^{q}+\frac{1-t}{2}|f'(b)|^{q}\right)dt\right)^{\frac{1}{q}}\right]$$

Now using the fact that $|A|^{p}\leq|p.A|$

$$|I| \leq \frac{(b-a)^{\xi}}{2^{\xi+1}\mathrm{B}(\xi)\Gamma(\xi)}\left[\left(\int_0^1 \left|p.\frac{2(1-t)^{\xi}-t^{\xi}}{3}\right| dt\right)^{\frac{1}{p}}\left(\int_0^1 \left(\frac{1+t}{2}|f'(b)|^q+\frac{1-t}{2}|f'(a)|^q\right)dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1 \left|p.\frac{t^{\xi}-2(1-t)^{\xi}}{3}\right| dt\right)^{\frac{1}{p}}\left(\int_0^1 \left(\frac{1+t}{2}|f'(a)|^q+\frac{1-t}{2}|f'(b)|^q\right)dt\right)^{\frac{1}{q}}\right]$$

By calculating the integrals that is in the above inequalities, we get desired result.

Theorem 3. Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) with $a<b$ and $f'\in L[a,b]$. If $|f'|^q$ is a convex function, we have the following inequality for Atangana-Baleanu fractional integral operators:

$$|I| \leq \frac{(b-a)^{\xi}}{2^{\xi+1}\mathrm{B}(\xi)\Gamma(\xi)}\left(\frac{-4\left(1-\frac{2^{\frac{1}{\xi}}}{2^{\frac{1}{\xi}}+1}\right)^{\xi+1}-2\left(\frac{2^{\frac{1}{\xi}}}{2^{\frac{1}{\xi}}+1}\right)^{\xi+1}+3}{3(\xi+1)}\right)^{1-\frac{1}{q}}\left(\left(K_1|f'(b)|^q\right.\right.$$
$$\left.+K_2|f'(a)|^q\right)^{\frac{1}{q}}+\left(K_1|f'(a)|^q+K_2|f'(b)|^q\right)^{\frac{1}{q}}\Big)$$

and

$$K_1=\frac{-4\left(1+\frac{2^{\frac{1}{\xi}}}{2^{\frac{1}{\xi}}+1}\right)\left(1-\frac{2^{\frac{1}{\xi}}}{2^{\frac{1}{\xi}}+1}\right)^{\xi+1}-2\left(\frac{2^{\frac{1}{\xi}}}{2^{\frac{1}{\xi}}+1}\right)^{\xi+1}+3}{6(\xi+1)}+\frac{1-2\left(\frac{2^{\frac{1}{\xi}}}{2^{\frac{1}{\xi}}+1}\right)^{\xi+2}}{6(\xi+2)}$$
$$+\frac{1-2\left(1-\frac{2^{\frac{1}{\xi}}}{2^{\frac{1}{\xi}}+1}\right)^{\xi+2}}{3(\xi+1)(\xi+2)}$$

$$K_2=\frac{-4\left(1-\frac{2^{\frac{1}{\xi}}}{2^{\frac{1}{\xi}}+1}\right)^{\xi+2}+2\left(\frac{2^{\frac{1}{\xi}}}{2^{\frac{1}{\xi}}+1}\right)^{\xi+2}+1}{6(\xi+2)}+\frac{1-2\left(\frac{2^{\frac{1}{\xi}}}{2^{\frac{1}{\xi}}+1}\right)^{\xi+1}}{6(\xi+1)}$$

where $\xi\in(0,1]$, $q\geq 1$ and $\mathrm{B}(\xi)$ is the normalization function.

Proof. By Lemma 1, we get

$$|I| \leq \frac{(b-a)^{\xi}}{2^{\xi+1}\mathrm{B}(\xi)\Gamma(\xi)}\left(\int_0^1 \left|\frac{2(1-t)^{\xi}-t^{\xi}}{3}\right|\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|dt + \int_0^1 \left|\frac{t^{\xi}-2(1-t)^{\xi}}{3}\right|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt\right)$$

By applying power mean inequality, we get

$$|I| \leq \frac{(b-a)^{\xi}}{2^{\xi+1}\mathrm{B}(\xi)\Gamma(\xi)}\left[\left(\int_0^1 \left|\frac{2(1-t)^{\xi}-t^{\xi}}{3}\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1 \left|\frac{2(1-t)^{\xi}-t^{\xi}}{3}\right|\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q dt\right)^{\frac{1}{q}} + \left(\int_0^1 \left|\frac{t^{\xi}-2(1-t)^{\xi}}{3}\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1 \left|\frac{t^{\xi}-2(1-t)^{\xi}}{3}\right|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right]$$

By using convexity of $|f'|^q$, we obtain

$$|I| \leq \frac{(b-a)^{\xi}}{2^{\xi+1}\mathrm{B}(\xi)\Gamma(\xi)}\left[\left(\int_0^1 \left|\frac{2(1-t)^{\xi}-t^{\xi}}{3}\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1 \left|\frac{2(1-t)^{\xi}-t^{\xi}}{3}\right|\left(\frac{1+t}{2}|f'(b)|^q + \frac{1-t}{2}|f'(a)|^q\right)dt\right)^{\frac{1}{q}} + \left(\int_0^1 \left|\frac{t^{\xi}-2(1-t)^{\xi}}{3}\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1 \left|\frac{t^{\xi}-2(1-t)^{\xi}}{3}\right|\left(\frac{1+t}{2}|f'(a)|^q + \frac{1-t}{2}|f'(b)|^q\right)dt\right)^{\frac{1}{q}}\right]$$

By computing the above integrals, the proof is completed.

Theorem 4. Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) with $a<b$ and $f'\in L[a,b]$. If $|f'|^q$ is a convex function, we have the following inequality for Atangana-Baleanu fractional integral operators:

$$|I| \leq \frac{(b-a)^{\xi}}{2^{\xi+1}\mathrm{B}(\xi)\Gamma(\xi)}\left[\frac{6-4\left(\frac{2^{\frac{1}{\xi}}}{2^{\frac{1}{\xi}}+1}\right)^{\xi+1}-8\left(1-\frac{2^{\frac{1}{\xi}}}{2^{\frac{1}{\xi}}+1}\right)^{\xi+1}}{3(\xi+1)}+\frac{|f'(a)|^q+|f'(b)|^q}{q}\right]$$

where $p^{-1}+q^{-1}=1$, $\xi \in (0,1]$, $q>1$ and $\mathrm{B}(\xi)$ is the normalization function.

Proof. By using identity that is given in Lemma 1, we get

$$|I| \leq \frac{(b-a)^{\xi}}{2^{\xi+1}\mathrm{B}(\xi)\Gamma(\xi)}\left(\int_0^1\left|\frac{2(1-t)^{\xi}-t^{\xi}}{3}\right|\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|dt\right.$$
$$\left.+\int_0^1\left|\frac{t^{\xi}-2(1-t)^{\xi}}{3}\right|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt\right)$$

By using the Young inequality, we obtain

$$|I| \leq \frac{(b-a)^{\xi}}{2^{\xi+1}\mathrm{B}(\xi)\Gamma(\xi)}\int_0^1\left(\frac{\left|\frac{2(1-t)^{\xi}-t^{\xi}}{3}\right|^p}{p}+\frac{\left|f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^q}{q}\right)dt$$
$$+\frac{(b-a)^{\xi}}{2^{\xi+1}\mathrm{B}(\xi)\Gamma(\xi)}\int_0^1\left(\frac{\left|\frac{t^{\xi}-2(1-t)^{\xi}}{3}\right|^p}{p}+\frac{\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q}{q}\right)dt$$

By using convexity of $|f'|^q$, we have

$$|I| \leq \frac{(b-a)^{\xi}}{2^{\xi+1}\mathrm{B}(\xi)\Gamma(\xi)}\int_0^1\left(\frac{\left|\frac{2(1-t)^{\xi}-t^{\xi}}{3}\right|^p}{p}+\frac{\frac{1+t}{2}|f'(b)|^q+\frac{1-t}{2}|f'(a)|^q}{q}\right)dt$$
$$+\frac{(b-a)^{\xi}}{2^{\xi+1}\mathrm{B}(\xi)\Gamma(\xi)}\int_0^1\left(\frac{\left|\frac{t^{\xi}-2(1-t)^{\xi}}{3}\right|^p}{p}+\frac{\frac{1+t}{2}|f'(a)|^q+\frac{1-t}{2}|f'(b)|^q}{q}\right)dt$$

Now using the fact that $|A|^p \leq |p.A|$

$$|I| \leq \frac{(b-a)^{\xi}}{2^{\xi+1}\mathrm{B}(\xi)\Gamma(\xi)} \int_0^1 \left(\frac{\left| p.\frac{2(1-t)^{\xi}-t^{\xi}}{3} \right|}{p} + \frac{\frac{1+t}{2}|f'(b)|^q + \frac{1-t}{2}|f'(a)|^q}{q} \right) dt$$

$$+ \frac{(b-a)^{\xi}}{2^{\xi+1}\mathrm{B}(\xi)\Gamma(\xi)} \int_0^1 \left(\frac{\left| p.\frac{t^{\xi}-2(1-t)^{\xi}}{3} \right|}{p} + \frac{\frac{1+t}{2}|f'(a)|^q + \frac{1-t}{2}|f'(b)|^q}{q} \right) dt$$

By a simple computation, we have the desired result.

REFERENCES

[1] M. Alomari, M. Darus and S.S. Dragomir, New inequalities of Simpson's type for $s-$ convex functions with applications, *RGMIA Res. Rep. Coll.* 12 (4) **(2009)**.
[2] M. Alomari and M. Darus, On some inequalities of Simpson-type via quasi-convex functions and applications, *Transylvanian Journal of Mathematics and Mech.* 2 (1) **(2010)** 15–24.
[3] M.Z. Sarikaya, E. Set and M.E. Özdemir, On new inequalities of Simpson's type for $s-$ convex functions, *Computers & Mathematics with Appl.* 60 **(2010)** 2191–2199.
[4] N.Ujevic, Sharp inequalities of Simpson type and Ostrowski type, *Computers & Mathematics with Appl.* 48 **(2004)** 145-151.
[5] N.Ujevic, Two sharp inequalities of Simpson type and applications, *Georgian Mathematical Journal* 1 (11) **(2004)** 187–194.
[6] R.-F. Bai, F. Qi and B.-Y. Xi, Hermite-Hadamard type inequalities for the m - and (α, m) -logarithmically convex functions, *Filomat* 27 **(2013)** 1-7.
[7] S. S. Dragomir, R. P. Agarwal, and P. Cerone, On Simpson's inequality and applications, *Journal of Inequalities and Appl.* 5 **(2000)** 533–579.
[8] T. Abdeljawad & D. Baleanu, On Fractional Derivatives with Exponential Kernel and their Discrete Versions. *Reports on Mathematical Physics, 80*(1), **(2017)**, 11-27.

Some New Inequalities for Exponentially Quasi-Convex Functions on the Coordinates and Related Hadamard Type Integral Inequalities

Ahmet Ocak AKDEMİR[1], Sinan ASLAN[1] and Erhan SET[2]

[1]*Department of Mathematics, Faculty of Science and Arts,*
Ağrı İbrahim Çeçen University, Ağrı-TURKEY
[2]*Department of Mathematics, Faculty of Science and Arts,*
Ordu University, Ordu-TURKEY

aocakakdemir@gmail.com
sinanaslan0407@gmail.com
erhanset@yahoo.com

ABSTRACT

In this presentation, we recalled the notion of quasi-convex functions which have become a very popular topic in recent years and have been studied by many mathematicians. First, we have given the definiton of exponentially quasi-convex functions on the co-ordinates as a new concept. Then, we have proved some new Hermite-Hadamard type integral inequalities via exponentially quasi-convex functions on the coordinates.

INTRODUCTION

In [1], Dragomir mentions an expansion of the concept of convex function, which is used in many inequalities in the field of inequality theory and has applications in different fields of mathematics, especially convex programming.

Definition 1 *Let us consider the bidimensional interval* $\Delta=[a,b]\times c,d]$ *in* $\mathbb{R}^2$ *with* $a<b$, $c<d$. *A function* $f:\Delta\to\mathbb{R}$ *will be called convex on the co-ordinates if the partial mappings* $f_y:[a,b]\to\mathbb{R}$, $f_y(u)=f(u,y)$ *and* $f_x:[c,d]\to\mathbb{R}$, $f_x(v)=f(x,v)$ *are convex where defined for all* $y\in[c,d]$ *and* $x\in[a,b]$ *Recall that the mapping* $f:\Delta\to\mathbb{R}$ *is convex on* Δ *if the following inequality holds,*

$$f(\lambda x+(1-\lambda)z,\lambda y+(1-\lambda)w)\le\lambda f(x,y)+(1-\lambda)f(z,w)$$

for all $(x,y),(z,w)\in\Delta$ and $\lambda\in 0,1]$.

Expressing convex functions in coordinates brought up the question that it is possible for Hermite-Hadamard inequality to expand into coordinates. The answer to this motivating question has been found in Dragomir's paper (see [1]) and has taken its place in the literature as the expansion of Hermite-Hadamard inequality to a rectangle from the plane $\mathbb{R}^2$. stated below.

Theorem 1 *Suppose that* $f:\Delta=[a,b]\times c,d]\to \mathrm{R}$ *is convex on the co-ordinates on* Δ. *Then one has the inequalities;*

$$
\begin{aligned}
& f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\
& \leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right] \\
& \leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy \\
& \leq \frac{1}{4}\left[\frac{1}{(b-a)}\int_a^b f(x,c)dx+\frac{1}{(b-a)}\int_a^b f(x,d)dx\right. \\
& \left.+\frac{1}{(d-c)}\int_c^d f(a,y)dy+\frac{1}{(d-c)}\int_c^d f(b,y)dy\right] \\
& \leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.
\end{aligned} \tag{1.1}
$$

The above inequalities are sharp.

Numerous variants of this inequality were obtained for convexity and other types of convex functions in coordinates (See the papers [2-11]).

EXPONENTIALLY QUASI-CONVEX FUNCTIONS ON THE CO-ORDINATES

Definition 2 *Let us consider the bidimensional interval* $\Delta=[a,b]\times c,d]$ *in* R^2 *with* $a<b$ *and* $c<d$. *The mapping* $f:\Delta\to R$ *is exponential Quasi-convex function on the co-ordinates on* Δ, *if the following inequality holds,*

$$f(tx+(1-t)z,ty+(1-t)w)\leq \max\left\{\frac{f(x,y)}{e^{\alpha(x+y)}},\frac{f(z,w)}{e^{\alpha(z+w)}}\right\}$$

for all $(x,y),(z,w)\in\Delta,\alpha\in R$ and $t\in[0,1]$.

An equivalent definition of the exponential Quasi-convex function definition in co-ordinates can be done as follows:

Definition 3 *The mapping* $f:\Delta\to R$ *is exponential Quasi-convex on the co-ordinates on* Δ, *if the following inequality holds,*

$$f(ta+(1-t)b,sc+(1-s)d)\leq \max\left\{\frac{f(a,c)}{e^{\alpha(a+c)}},\frac{f(a,d)}{e^{\alpha(a+d)}},\frac{f(b,c)}{e^{\alpha(b+c)}},\frac{f(b,d)}{e^{\alpha(b+d)}}\right\}.$$

for all $(a,c),(a,d),(b,c),(b,d)\in\Delta,\alpha\in R$ and $t,s\in[0,1]$

Lemma 1 *A function* $f:\Delta\to R$ *will be called exponential Quasi-convex function on the co-ordinates on* Δ*, if the partial mappings* $f_y:[a,b]\to R$, $f_y(u)=e^{\alpha y}f(u,y)$ *and* $f_x:[c,d]\to R$, $f_x(v)=e^{\alpha x}f(x,v)$ *are exponential Quasi-convex function on the co-ordinates on* Δ*, where defined for all* $y\in c,d]$ *and* $x\in a,b]$.

Proof. From the definition of partial mapping f_x, we can write

$$f_x(tv_1+(1-t)v_2)=e^{\alpha x}f(x,tv_1+(1-t)v_2)$$

$$=e^{\alpha x}f(tx+(1-t)x,tv_1+(1-t)v_2)$$

$$\leq e^{\alpha x}\left[\max\left\{\frac{f(x,v_1)}{e^{\alpha(x+v_1)}},\frac{f(x,v_2)}{e^{\alpha(x+v_2)}}\right\}\right]$$

$$=\max\left\{\frac{f(x,v_1)}{e^{\alpha v_1}},\frac{f(x,v_2)}{e^{\alpha v_2}}\right\}$$

$$=\max\left\{\frac{f_x(v_1)}{e^{\alpha v_1}},\frac{f_x(v_2)}{e^{\alpha v_2}}\right\}$$

Similarly,

$$f_y(tu_1+(1-t)u_2)=e^{\alpha y}f(tu_1+(1-t)u_2,y)$$

$$=e^{\alpha y}f(tu_1+(1-t)u_2,ty+(1-t)y)$$

$$\leq e^{\alpha y}\left[\max\left\{\frac{f(u_1,y)}{e^{\alpha(u_1+y)}},\frac{f(u_2,y)}{e^{\alpha(u_2+y)}}\right\}\right]$$

$$=\max\left\{\frac{f(u_1,y)}{e^{\alpha u_1}},\frac{f(u_2,y)}{e^{\alpha u_2}}\right\}$$

$$=\max\left\{\frac{f_y(u_1)}{e^{\alpha u_1}},\frac{f_y(u_2)}{e^{\alpha u_2}}\right\}.$$

Proof is completed.

Theorem 2 *Let* $f:\Delta=[a,b]\times c,d]\to R$ *be partial differentiable mapping on* $\Delta=[a,b]\times c,d]$ *and* $f\in L(\Delta)$, $\alpha\in R$*. If* f *is exponential Quasi-convex function on the co-ordinates on* Δ*, then the following inequality holds;*

$$\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy\leq\max\left\{\frac{f(a,c)}{e^{\alpha(a+c)}},\frac{f(a,d)}{e^{\alpha(a+d)}},\frac{f(b,c)}{e^{\alpha(b+c)}},\frac{f(b,d)}{e^{\alpha(b+d)}}\right\}.$$

Proof. By the definition of the exponential Quasi-convex functions on the co-ordinates on Δ, we can write

$$f\left(ta+(1-t)b, sc+(1-s)d\right) \le \max\left\{\frac{f(a,c)}{e^{\alpha(a+c)}}, \frac{f(a,d)}{e^{\alpha(a+d)}}, \frac{f(b,c)}{e^{\alpha(b+c)}}, \frac{f(b,d)}{e^{\alpha(b+d)}}\right\}.$$

By integrating both sides of the above inequality with respect to t,s on $[0,1]^2$, we have

$$\int_0^1\int_0^1 f\left(ta+(1-t)b, sc+(1-s)d\right)dtds$$

$$\le \int_0^1\int_0^1 \max\left\{\frac{f(a,c)}{e^{\alpha(a+c)}}, \frac{f(a,d)}{e^{\alpha(a+d)}}, \frac{f(b,c)}{e^{\alpha(b+c)}}, \frac{f(b,d)}{e^{\alpha(b+d)}}\right\}dtds.$$

By computing the above integrals, we obtain the desired result.

Theorem 3 *Let* $f:\Delta=[a,b]\times c,d]\to R$ *be partial differentiable mapping on* $\Delta=[a,b]\times c,d]$ *and* $f\in L(\Delta), \alpha\in R.$ *If* $|f|$ *is exponential Quasi-convex function on the co-ordinates on* Δ, $p,q>1, \frac{1}{p}+\frac{1}{q}=1,$ *then the following inequality holds;*

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy\right|$$

$$\le \frac{1}{p}+\frac{1}{q}\max\left\{\frac{|f(a,c)|^q}{e^{\alpha q(a+c)}}, \frac{|f(a,d)|^q}{e^{\alpha q(a+d)}}, \frac{|f(b,c)|^q}{e^{\alpha q(b+c)}}, \frac{|f(b,d)|^q}{e^{\alpha q(b+d)}}\right\}.$$

Proof. By the definition of the exponential Quasi-convex functions on the co-ordinates on Δ, we can write

$$f\left(ta+(1-t)b, sc+(1-s)d\right)$$

$$\le \max\left\{\frac{f(a,c)}{e^{\alpha(a+c)}}, \frac{f(a,d)}{e^{\alpha(a+d)}}, \frac{f(b,c)}{e^{\alpha(b+c)}}, \frac{f(b,d)}{e^{\alpha(b+d)}}\right\}.$$

If the absolute value property is used in integral and by integrating both sides of the above inequality with respect to t,s on $[0,1]^2$, we can write

$$\left|\int_0^1\int_0^1 f\left(ta+(1-t)b, sc+(1-s)d\right)dtds\right|$$

$$\le \int_0^1\int_0^1 \left|\max\left\{\frac{f(a,c)}{e^{\alpha(a+c)}}, \frac{f(a,d)}{e^{\alpha(a+d)}}, \frac{f(b,c)}{e^{\alpha(b+c)}}, \frac{f(b,d)}{e^{\alpha(b+d)}}\right\}\right|dtds$$

$$\le \int_0^1\int_0^1 \max\left\{\left|\frac{f(a,c)}{e^{\alpha(a+c)}}\right|, \left|\frac{f(a,d)}{e^{\alpha(a+d)}}\right|, \left|\frac{f(b,c)}{e^{\alpha(b+c)}}\right|, \left|\frac{f(b,d)}{e^{\alpha(b+d)}}\right|\right\}dtds.$$

If we apply the Young's inequality to the right-hand side of the inequality, we get

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy\right|$$

$$\leq \frac{1}{p}\left(\int_0^1\int_0^1 dtds\right)+\frac{1}{q}\left(\int_0^1\int_0^1\left|\max\left\{\left|\frac{f(a,c)}{e^{\alpha(a+c)}}\right|,\left|\frac{f(a,d)}{e^{\alpha(a+d)}}\right|,\left|\frac{f(b,c)}{e^{\alpha(b+c)}}\right|,\left|\frac{f(b,d)}{e^{\alpha(b+d)}}\right|\right\}\right|^q dtds\right).$$

By computing the above integrals, we obtain the desired result.

Proposition 1 *If* $f:\Delta\to R$ *is exponential Quasi-convex functions on the co-ordinates on* Δ *and* $k\geq 0$ *then* kf *is exponential Quasi-convex functions on the co-ordinates on* Δ.

Proof. By the definition of the exponential Quasi-convex functions on the co-ordinates on Δ, we can write

$$f(ta+(1-t)b, sc+(1-s)d)$$

$$\leq \max\left\{\frac{f(a,c)}{e^{\alpha(a+c)}},\frac{f(a,d)}{e^{\alpha(a+d)}},\frac{f(b,c)}{e^{\alpha(b+c)}},\frac{f(b,d)}{e^{\alpha(b+d)}}\right\}.$$

If both sides are multiplied by k, we have,

$$(kf)(ta+(1-t)b, sc+(1-s)d)$$

$$\leq \max\left\{\frac{kf(a,c)}{e^{\alpha(a+c)}},\frac{kf(a,d)}{e^{\alpha(a+d)}},\frac{kf(b,c)}{e^{\alpha(b+c)}},\frac{kf(b,d)}{e^{\alpha(b+d)}}\right\}.$$

Therefore (kf) is exponential Quasi-convex functions on the co-ordinates on Δ.

ACKNOWLEDGEMENTS

This study was supported by Ağrı İbrahim Çeçen University BAP Coordination Unit with the Project number FEF.22.004.

REFERENCES

[1] S.S. Dragomir, On Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane, Taiwanese Journal of Math., 5, 2001, 775-788.
[2] M.E. Özdemir, M.A. Latif, A.O. Akdemir, On some Hadamard-type inequalities for product of two $s-$convex functions on the co-ordinates, Journal of Inequalities and Applications , 2012, February-2012, DOI: 10.1186/1029-242X-2012-21.
[3] E. Set, M.Z. Sarikaya, A.O. Akdemir, A new general inequality for double integrals, American Institu of Phsyics (AIP) Conference Proceedings , 1470, 122-125 pp., October-2012, DOI: http://dx.doi.org/10.1063/1.4747655.
[4] M. Alomari and M. Darus, Hadamard-Type inequalities for $s-$convex functions, International Mathematical Forum, (3) 2008, no. 40, 1965 - 1975.

[5] M.E. Özdemir, A.O. Akdemir, On the Hadamard-type inequalities involving product of two convex functions on the co-ordinates, Tamkang Journal of Mathematics, vol. 46, no. 2, pp. 129-142, 2015.
[6] A.O. Akdemir, M.E. Özdemir, Some Hadamard-type inequalities for co-ordinated $P-$ convex functions and Godunova-Levin functions, American Institu of Phsyics (AIP) Conference Proceedings , 1309, 7-15 pp., 2010.
[7] M. E. Özdemir, M.A. Latif, A.O. Akdemir, On some Hadamard-type inequalities for product of two $h-$convex functions on the co-ordinates, Turkish Journal of Science, 1 (1), 2016, 48-58.
[8] M.K. Bakula and J. Pecaric, On the Jensen's inequality for convex functions on the co-ordinates in a rectangle from the plane, Taiwanese Journal of Math., 5, 2006, 1271-1292.
[9] M.E. Özdemir, A.O. Akdemir, C. Yıldız, On co-ordinated Quasi-convex functions, Czechoslovak Mathematical Journal , 62-4, 889-900 pp., 2012.
[10] M. Alomari and M. Darus, The Hadamard's inequality for $s-$convex functions of $2-$ variables, Int. Journal of Math. Analysis, 2(13), 2008, 629-638.
[11] M.E. Özdemir, E. Set and M.Z. Sarıkaya, Some new Hadamard's type inequalities for co-ordinated $m-$convex and $(\alpha,m)-$convex functions, Hacettepe J. of. Math. and St., 40, 219-229, (2011).

New Integral Inequalities Involving the Proportional Caputo-Hybrid Operators for *s*-Convex Functions

Barış ÇELİK[1], Erhan SET[1] and Ahmet Ocak AKDEMİR[2]

[1]*Department of Mathematics, Faculty of Sciences and Arts, Ordu University, Ordu, TURKEY*
[2]*Department of Mathematics, Faculty of Sciences and Arts, Ağrı İbrahim Çeçen University, Ağrı, TURKEY*

bariscelik15@hotmail.com
erhanset@yahoo.com
aocakakdemir@gmail.com

ABSTRACT

In this paper, we establish some new Hermite-Hadamard-type inequalities for s-convex functions in the second sense via the proportional Caputo-hybrid operators. Hölder and Young's inequalities were used to prove the new results obtained. In addition, it is seen that the results obtained are reduced to the results obtained previously in the literature.

1. INTRODUCTION

In this section, we present the preliminaries and definitions.

Definition 1 *[1] A function* $f:[0,\infty)\to\mathbb{R}$, *is said to be* s*-convex in the second sense if*

$$f\big(ta+(1-t)b\big)\le t^s f(a)+(1-t)^s f(b)$$

for all $a,b\in[0,\infty)$, $t\in 0,1]$ and for some fixed $s\in(0,1]$.

Besides, the concept of convex function has many useful properties, it also forms the basis of the Hermite-Hadamard (HH) inequality, one of the well-known fundamental and famous inequalities in the literature. The HH inequality, which has the potential to produce lower and upper bounds to the mean value of a convex function in the Cauchy sense, has inspired many researchers in mathematical analysis with its applications. The statement of this inequality is as follows.
If a mapping $f:I\subseteq\mathbb{R}\to\mathbb{R}$ is a convex function on I and $a,b\in I$, $a<b$, then

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)dx\le\frac{f(a)+f(b)}{2}.$$

We recommend that readers refer to papers [2, 3, 4, 5, 6] for versions of the HH inequality for different kinds of convex functions, its modification to co-ordinates, and its expansions with the help of various new fractional integral operators.

The proportional Caputo hybrid operator, which was put forward as a non-local and singular operator containing both derivative and integral operator parts in its definition, and which is a simple linear combination of the Riemann-Liouville integral and the Caputo derivative operators, is defined as follows (see [7]).

Definition 3 *Let* $f : I \subseteq \mathbb{R}^+ \to \mathbb{R}$ *be a differentiable function on* I°. *Also let* $f, f' \in L_1$ *are functions on I. Then, the proportional Caputo-hybrid operator may be defined as*

$$ {}_0^{CPC}D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \left(K_1(\alpha) f(\tau) + K_0(\alpha) f'(\tau) \right) (t-\tau)^{-\alpha} d\tau $$

where $\alpha \in [0,1]$ and K_0 and K_1 are functions satisfying

$$ \lim_{\alpha \to 0^+} K_0(\alpha) = 0; \quad \lim_{\alpha \to 1^-} K_0(\alpha) = 1; \quad K_0(\alpha) \neq 0, \quad \alpha \in (0,1]; \tag{1.1} $$

$$ \lim_{\alpha \to 0^+} K_1(\alpha) = 0; \quad \lim_{\alpha \to 1^-} K_1(\alpha) = 1; \quad K_1(\alpha) \neq 0, \quad \alpha \in [0,1). \tag{1.2} $$

Remark 1 (See [7]**)***We originally wrote this paper using the specific case*

$$ K_0(\alpha, t) = \alpha t^{1-\alpha} $$
$$ K_1(\alpha, t) = (1-\alpha) t^\alpha $$

which is afforded special attention in [8].

In [9], Gürbüz et al. established following identity for convex functions:

Lemma 1 *Let* $f : I \subseteq \mathbb{R}^+ \to \mathbb{R}$ *be a twice differentiable function on* I°. *Also let* f *and* f' *are* L^1 *functions on* I. *Then, the following equality holds:*

$$ \begin{aligned} & K_1(\alpha) \int_0^1 t^{1-\alpha} f'(ta + (1-t)x) dt + K_0(\alpha) \int_0^1 t^{1-\alpha} f''(ta + (1-t)x) dt \\ & + K_1(\alpha) \int_0^1 t^{1-\alpha} f'(tx + (1-t)b) dt + K_0(\alpha) \int_0^1 t^{1-\alpha} f''(tx + (1-t)b) dt \\ & = -\frac{K_1(\alpha) f(a) + K_0(\alpha) f'(a)}{x-a} - \frac{K_1(\alpha) f(x) + K_0(\alpha) f'(x)}{b-x} \\ & + \Gamma(2-\alpha) \left(\frac{{}_a^{CPC}D_x^\alpha f(x)}{(x-a)^{2-\alpha}} + \frac{{}_x^{CPC}D_b^\alpha f(b)}{(b-x)^{2-\alpha}} \right) \end{aligned} $$

where $\alpha \in [0,1]$, $a < x < b$ and K_0 and K_1 are functions satisfing the conditions (1.1) and (1.2).

The main purpose of this article is to prove new integral inequalities for the class of s-convex functions that are differentiable with the help of proportional-Caputo hybrid operators and the identity in [9] found in the literature.

2. MAIN RESULTS

Theorem 1 *Let $f : I \subseteq \mathbb{R}^+ \to \mathbb{R}$ be a twice differentiable function on I°. Also let $f, f' \in L_1$ are functions on I. If f' and f'' are s-convex in the second sense on I, then the following inequality holds:*

$$\begin{aligned}
&\Bigg| -\frac{K_1(\alpha)f(a)+K_0(\alpha)f'(a)}{x-a} - \frac{K_1(\alpha)f(x)+K_0(\alpha)f'(x)}{b-x} \\
&+\Gamma(2-\alpha)\left(\frac{{}_a^{CPC}D_x^\alpha f(x)}{(x-a)^{2-\alpha}} + \frac{{}_x^{CPC}D_b^\alpha f(b)}{(b-x)^{2-\alpha}}\right)\Bigg| \\
&\le \frac{K_1(\alpha)|f'(a)| + K_0(\alpha)|f''(a)|}{2+s-\alpha} \\
&+\left(K_1(\alpha)|f'(b)| + K_0(\alpha)|f''(b)|\right)\left(\beta(2-\alpha, s+1) + \frac{1}{2+s-\alpha}\right) \\
&+\left(K_1(\alpha)|f'(x)| + K_0(\alpha)|f''(x)|\right)\beta(2-\alpha, s+1)
\end{aligned}$$

where $\alpha \in [0,1]$, $s \in (0,1]$ $a < x < b$ and K_0 and K_1 are functions satisfying the conditions (1.1) and (1.2).

Proof. From Lemma 1 and using properties of absolute value, we have

$$\begin{aligned}
&\Bigg| -\frac{K_1(\alpha)f(a)+K_0(\alpha)f'(a)}{x-a} - \frac{K_1(\alpha)f(x)+K_0(\alpha)f'(x)}{b-x} \\
&+\Gamma(2-\alpha)\left(\frac{{}_a^{CPC}D_x^\alpha f(x)}{(x-a)^{2-\alpha}} + \frac{{}_x^{CPC}D_b^\alpha f(b)}{(b-x)^{2-\alpha}}\right)\Bigg| \\
&\le K_1(\alpha)\int_0^1 t^{1-\alpha}\left|f'(ta+(1-t)x)\right|dt \\
&+K_0(\alpha)\int_0^1 t^{1-\alpha}\left|f''(ta+(1-t)x)\right|dt \\
&+K_1(\alpha)\int_0^1 t^{1-\alpha}\left|f'(tx+(1-t)b)\right|dt
\end{aligned}$$

$$+K_0(\alpha)\int_0^1 t^{1-\alpha}\left|f''(tx+(1-t)b)\right|dt. \tag{2.1}$$

As f' and f'' are s-convex functions on I, we can write

$$\left|-\frac{K_1(\alpha)f(a)+K_0(\alpha)f'(a)}{x-a}-\frac{K_1(\alpha)f(x)+K_0(\alpha)f'(x)}{b-x}\right.$$
$$\left.+\Gamma(2-\alpha)\left(\frac{{}_a^{CPC}D_x^{\alpha}f(x)}{(x-a)^{2-\alpha}}+\frac{{}_x^{CPC}D_b^{\alpha}f(b)}{(b-x)^{2-\alpha}}\right)\right|$$
$$\leq K_1(\alpha)\int_0^1 t^{1-\alpha}\left(t^s\left|f'(a)\right|+(1-t)^s\left|f'(x)\right|\right)dt$$
$$+K_0(\alpha)\int_0^1 t^{1-\alpha}\left(t^s\left|f''(a)\right|+(1-t)^s\left|f''(x)\right|\right)dt$$
$$+K_1(\alpha)\int_0^1 t^{1-\alpha}\left(t^s\left|f'(x)\right|+(1-t)^s\left|f'(b)\right|\right)dt$$
$$+K_0(\alpha)\int_0^1 t^{1-\alpha}\left(t^s\left|f''(x)\right|+(1-t)^s\left|f''(b)\right|\right)dt$$
$$=\left(K_1(\alpha)\left|f'(x)\right|+K_0(\alpha)\left|f''(x)\right|\right)\int_0^1\left(t^{1-\alpha}(1-t)^s+t^{1+s-\alpha}\right)dt$$
$$+\left(K_1(\alpha)\left|f'(a)\right|+K_0(\alpha)\left|f''(a)\right|\right)\int_0^1 t^{1+s-\alpha}dt$$
$$+\left(K_1(\alpha)\left|f'(b)\right|+K_0(\alpha)\left|f''(b)\right|\right)\int_0^1 t^{1-\alpha}(1-t)^s dt$$

The proof is completed by making the necessary calculations.

Theorem 2 *Let $f: I\subseteq \mathbb{R}^+\to\mathbb{R}$ be a twice differentiable function on I°. Also let $f, f'\in L_1$ are functions on I. If $\left|f'\right|^q$ and $\left|f''\right|^q$ are s-convex in the second sense on I, then the following inequality holds:*

$$\left|-\frac{K_1(\alpha)f(a)+K_0(\alpha)f'(a)}{x-a}-\frac{K_1(\alpha)f(x)+K_0(\alpha)f'(x)}{b-x}\right.$$
$$\left.+\Gamma(2-\alpha)\left(\frac{{}_a^{CPC}D_x^{\alpha}f(x)}{(x-a)^{2-\alpha}}+\frac{{}_x^{CPC}D_b^{\alpha}f(b)}{(b-x)^{2-\alpha}}\right)\right|$$
$$\leq\frac{1}{((1-\alpha)p+1)^{\frac{1}{p}}(s+1)^{\frac{1}{q}}}\left[K_1(\alpha)\left(\left|f'(a)\right|^q+\left|f'(x)\right|^q\right)^{\frac{1}{q}}\right.$$

$$+K_0(\alpha)\left(\left|f''(a)\right|^q+\left|f''(x)\right|^q\right)^{\frac{1}{q}}$$
$$+K_1(\alpha)\left(\left|f'(x)\right|^q+\left|f'(b)\right|^q\right)^{\frac{1}{q}}$$
$$\left.+K_0(\alpha)\left(\left|f''(x)\right|^q+\left|f''(b)\right|^q\right)^{\frac{1}{q}}\right]$$

where $\alpha\in[0,1)$, $s\in(0,1]$, $a<x<b$, $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and K_0 and K_1 are functions satisfying the conditions (1.1) and (1.2).

Proof. By applying Hölder's inequality to (2.1), we get

$$\left|-\frac{K_1(\alpha)f(a)+K_0(\alpha)f'(a)}{x-a}-\frac{K_1(\alpha)f(x)+K_0(\alpha)f'(x)}{b-x}\right.$$
$$\left.+\Gamma(2-\alpha)\left(\frac{{}_a^{CPC}D_x^{\alpha}f(x)}{(x-a)^{2-\alpha}}+\frac{{}_x^{CPC}D_b^{\alpha}f(b)}{(b-x)^{2-\alpha}}\right)\right|$$
$$\le K_1(\alpha)\left[\left(\int_0^1 t^{(1-\alpha)p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'(ta+(1-t)x)\right|^q dt\right)^{\frac{1}{q}}\right]$$
$$+K_0(\alpha)\left[\left(\int_0^1 t^{(1-\alpha)p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f''(ta+(1-t)x)\right|^q dt\right)^{\frac{1}{q}}\right]$$
$$+K_1(\alpha)\left[\left(\int_0^1 t^{(1-\alpha)p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right]$$
$$+K_0(\alpha)\left[\left(\int_0^1 t^{(1-\alpha)p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f''(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right].$$

Using s-convexity of $\left|f'\right|^q$ and $\left|f''\right|^q$, we have

$$\left|-\frac{K_1(\alpha)f(a)+K_0(\alpha)f'(a)}{x-a}-\frac{K_1(\alpha)f(x)+K_0(\alpha)f'(x)}{b-x}\right.$$
$$\left.+\Gamma(2-\alpha)\left(\frac{{}_a^{CPC}D_x^{\alpha}f(x)}{(x-a)^{2-\alpha}}+\frac{{}_x^{CPC}D_b^{\alpha}f(b)}{(b-x)^{2-\alpha}}\right)\right|$$

$$\leq \frac{1}{((1-\alpha)p+1)^{\frac{1}{p}}}\left\{K_1(\alpha)\left(\int_0^1\left(t^s\left|f'(a)\right|^q+(1-t)^s\left|f'(x)\right|^q\right)dt\right)^{\frac{1}{q}}\right.$$

$$+K_0(\alpha)\left(\int_0^1\left(t^s\left|f''(a)\right|^q+(1-t)^s\left|f''(x)\right|^q\right)dt\right)^{\frac{1}{q}}$$

$$+K_1(\alpha)\left(\int_0^1\left(t^s\left|f'(x)\right|^q+(1-t)^s\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}$$

$$\left.+K_0(\alpha)\left(\int_0^1\left(t^s\left|f''(x)\right|^q+(1-t)^s\left|f''(b)\right|^q\right)dt\right)^{\frac{1}{q}}\right\}.$$

With simple calculations, we get

$$\left|-\frac{K_1(\alpha)f(a)+K_0(\alpha)f'(a)}{x-a}-\frac{K_1(\alpha)f(x)+K_0(\alpha)f'(x)}{b-x}\right.$$

$$\left.+\Gamma(2-\alpha)\left(\frac{{}^{CPC}_{\ a}D^{\alpha}_{x}f(x)}{(x-a)^{2-\alpha}}+\frac{{}^{CPC}_{\ x}D^{\alpha}_{b}f(b)}{(b-x)^{2-\alpha}}\right)\right|$$

$$\leq \frac{1}{((1-\alpha)p+1)^{\frac{1}{p}}}\left\{K_1(\alpha)\left(\frac{\left|f'(a)\right|^q+\left|f'(x)\right|^q}{s+1}\right)^{\frac{1}{q}}\right.$$

$$+K_0(\alpha)\left(\frac{\left|f''(a)\right|^q+\left|f''(x)\right|^q}{s+1}\right)^{\frac{1}{q}}+K_1(\alpha)\left(\frac{\left|f'(x)\right|^q+\left|f'(b)\right|^q}{s+1}\right)^{\frac{1}{q}}$$

$$\left.+K_0(\alpha)\left(\frac{\left|f''(x)\right|^q+\left|f''(b)\right|^q}{s+1}\right)^{\frac{1}{q}}\right\}$$

which is the desired result.

Theorem 3 *Let* $f: I \subseteq \mathbb{R}^{+} \to \mathbb{R}$ *be a twice differentiable function on* I°. *Also let* $f, f' \in L_1$ *are functions on* I. *If* $\left|f'\right|^q$ *and* $\left|f''\right|^q$ *are* s*-convex in the second sense on* I*, then the following inequality holds:*

$$\left|-\frac{K_1(\alpha)f(a)+K_0(\alpha)f'(a)}{x-a}-\frac{K_1(\alpha)f(x)+K_0(\alpha)f'(x)}{b-x}\right.$$

$$+\Gamma(2-\alpha)\left(\frac{{}^{CPC}_{\ a}D_x^{\alpha} f(x)}{(x-a)^{2-\alpha}}+\frac{{}^{CPC}_{\ x}D_b^{\alpha} f(b)}{(b-x)^{2-\alpha}}\right)\Bigg|\times(2-\alpha)^{1-\frac{1}{q}}$$

$$\leq\left\{K_1(\alpha)\left(\frac{|f'(a)|^q}{2+s-\alpha}+|f'(x)|^q\beta(2-\alpha,s+1)\right)^{\frac{1}{q}}\right.$$

$$+K_0(\alpha)\left(\frac{|f''(a)|^q}{2+s-\alpha}+|f''(x)|^q\beta(2-\alpha,s+1)\right)^{\frac{1}{q}}$$

$$+K_1(\alpha)\left(\frac{|f'(x)|^q}{2+s-\alpha}+|f'(b)|^q\beta(2-\alpha,s+1)\right)^{\frac{1}{q}}$$

$$\left.+K_0(\alpha)\left(\frac{|f''(x)|^q}{2+s-\alpha}+|f''(b)|^q\beta(2-\alpha,s+1)\right)^{\frac{1}{q}}\right\}$$

where $\alpha\in(0,1]$, $s\in(0,1]$, $a<x<b$, $q\geq 1$ and K_0 and K_1 are functions satisfying the conditions (1.1) and (1.2).

Proof. By applying power mean inequality to (2.1), we get

$$\Bigg|-\frac{K_1(\alpha)f(a)+K_0(\alpha)f'(a)}{x-a}-\frac{K_1(\alpha)f(x)+K_0(\alpha)f'(x)}{b-x}$$

$$+\Gamma(2-\alpha)\left(\frac{{}^{CPC}_{\ a}D_x^{\alpha} f(x)}{(x-a)^{2-\alpha}}+\frac{{}^{CPC}_{\ x}D_b^{\alpha} f(b)}{(b-x)^{2-\alpha}}\right)\Bigg|$$

$$\leq K_1(\alpha)\left(\int_0^1 t^{1-\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{1-\alpha}|f'(ta+(1-t)x)|^q dt\right)^{\frac{1}{q}}$$

$$+K_0(\alpha)\left(\int_0^1 t^{1-\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{1-\alpha}|f''(ta+(1-t)x)|^q dt\right)^{\frac{1}{q}}$$

$$+K_1(\alpha)\left(\int_0^1 t^{1-\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{1-\alpha}|f'(tx+(1-t)b)|^q dt\right)^{\frac{1}{q}}$$

$$+K_0(\alpha)\left(\int_0^1 t^{1-\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{1-\alpha}|f''(tx+(1-t)b)|^q dt\right)^{\frac{1}{q}}.$$

Using s-convexity of $\left|f'\right|^q$ and $\left|f''\right|^q$, we have

$$\left| -\frac{K_1(\alpha)f(a)+K_0(\alpha)f'(a)}{x-a}-\frac{K_1(\alpha)f(x)+K_0(\alpha)f'(x)}{b-x} \right.$$
$$\left. +\Gamma(2-\alpha)\left(\frac{{}_{a}^{CPC}D_x^{\alpha} f(x)}{(x-a)^{2-\alpha}}+\frac{{}_{x}^{CPC}D_b^{\alpha} f(b)}{(b-x)^{2-\alpha}}\right)\right|$$
$$\leq\left(\frac{1}{2-\alpha}\right)^{1-\frac{1}{q}}\left\{K_1(\alpha)\left(\int_0^1 t^{1-\alpha}\left(t^s\left|f'(a)\right|^q+(1-t)^s\left|f'(x)\right|^q\right)dt\right)^{\frac{1}{q}}\right.$$
$$+K_0(\alpha)\left(\int_0^1 t^{1-\alpha}\left(t^s\left|f''(a)\right|^q+(1-t)^s\left|f''(x)\right|^q\right)dt\right)^{\frac{1}{q}}$$
$$+K_1(\alpha)\left(\int_0^1 t^{1-\alpha}\left(t^s\left|f'(x)\right|^q+(1-t)^s\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}$$
$$\left.+K_0(\alpha)\left(\int_0^1 t^{1-\alpha}\left(t^s\left|f''(x)\right|^q+(1-t)^s\left|f''(b)\right|^q\right)dt\right)^{\frac{1}{q}}\right\}.$$

Making necessary calculations, we get

$$\left| -\frac{K_1(\alpha)f(a)+K_0(\alpha)f'(a)}{x-a}-\frac{K_1(\alpha)f(x)+K_0(\alpha)f'(x)}{b-x} \right.$$
$$\left. +\Gamma(2-\alpha)\left(\frac{{}_{a}^{CPC}D_x^{\alpha} f(x)}{(x-a)^{2-\alpha}}+\frac{{}_{x}^{CPC}D_b^{\alpha} f(b)}{(b-x)^{2-\alpha}}\right)\right|$$
$$\leq\left(\frac{1}{2-\alpha}\right)^{1-\frac{1}{q}}\left\{K_1(\alpha)\left(\frac{\left|f'(a)\right|^q}{2+s-\alpha}+\left|f'(x)\right|^q\beta(2-\alpha,s+1)\right)^{\frac{1}{q}}\right.$$
$$+K_0(\alpha)\left(\frac{\left|f''(a)\right|^q}{2+s-\alpha}+\left|f''(x)\right|^q\beta(2-\alpha,s+1)\right)^{\frac{1}{q}}$$
$$+K_1(\alpha)\left(\frac{\left|f'(x)\right|^q}{2+s-\alpha}+\left|f'(b)\right|^q\beta(2-\alpha,s+1)\right)^{\frac{1}{q}}$$

$$+K_0(\alpha)\left(\frac{|f''(x)|^q}{2+s-\alpha}+|f''(b)|^q\beta(2-\alpha,s+1)\right)^{\frac{1}{q}}\Bigg\}.$$

So, the proof is completed.

Theorem 4 *Let* $f: I \subseteq \mathbb{R}^+ \to \mathbb{R}$ *be a twice differentiable function on* I°. *Also let* $f, f' \in L_1$ *are functions on* I. *If* $|f'|^q$ *and* $|f''|^q$ *are* s*-convex in the second sense on* I, *then the following inequality holds:*

$$\Bigg|-\frac{K_1(\alpha)f(a)+K_0(\alpha)f'(a)}{x-a}-\frac{K_1(\alpha)f(x)+K_0(\alpha)f'(x)}{b-x}$$
$$+\Gamma(2-\alpha)\left(\frac{{}_{a}^{CPC}D_x^\alpha f(x)}{(x-a)^{2-\alpha}}+\frac{{}_{x}^{CPC}D_b^\alpha f(b)}{(b-x)^{2-\alpha}}\right)\Bigg|$$
$$\leq \frac{2(K_1(\alpha)+K_0(\alpha))}{p^2(1-\alpha)+p}+\frac{K_1(\alpha)}{(s+1)q}\left(|f'(a)|^q+2|f'(x)|^q+|f'(b)|^q\right)$$
$$+\frac{K_0(\alpha)}{(s+1)q}\left(|f''(a)|^q+2|f''(x)|^q+|f''(b)|^q\right)$$

where $\alpha \in (0,1]$, $s \in (0,1]$, $a<x<b$, $\frac{1}{p}+\frac{1}{q}=1$, $q>1$ and K_0 and K_1 are functions satisfying (1.1) and (1.2).

Proof. Taking into account the Young inequality as $mn \leq \frac{m^p}{p}+\frac{n^q}{q}$ in (2.1), we get

$$\Bigg|-\frac{K_1(\alpha)f(a)+K_0(\alpha)f'(a)}{x-a}-\frac{K_1(\alpha)f(x)+K_0(\alpha)f'(x)}{b-x}$$
$$+\Gamma(2-\alpha)\left(\frac{{}_{a}^{CPC}D_x^\alpha f(x)}{(x-a)^{2-\alpha}}+\frac{{}_{x}^{CPC}D_b^\alpha f(b)}{(b-x)^{2-\alpha}}\right)\Bigg|$$
$$\leq K_1(\alpha)\left[\frac{1}{p}\int_0^1 t^{p(1-\alpha)}dt+\frac{1}{q}\int_0^1 |f'(ta+(1-t)x)|^q dt\right]$$
$$+K_0(\alpha)\left[\frac{1}{p}\int_0^1 t^{p(1-\alpha)}dt+\frac{1}{q}\int_0^1 |f''(ta+(1-t)x)|^q dt\right]$$
$$+K_1(\alpha)\left[\frac{1}{p}\int_0^1 t^{p(1-\alpha)}dt+\frac{1}{q}\int_0^1 |f'(tx+(1-t)b)|^q dt\right]$$

$$+K_0(\alpha)\left[\frac{1}{p}\int_0^1 t^{p(1-\alpha)}dt+\frac{1}{q}\int_0^1\left|f''(tx+(1-t)b)\right|^q dt\right].$$

Using $s-$convexity of $\left|f'\right|^q$ and $\left|f''\right|^q$, we have

$$\begin{aligned}&\left|-\frac{K_1(\alpha)f(a)+K_0(\alpha)f'(a)}{x-a}-\frac{K_1(\alpha)f(x)+K_0(\alpha)f'(x)}{b-x}\right.\\&\left.+\Gamma(2-\alpha)\left(\frac{{}_a^{CPC}D_x^\alpha f(x)}{(x-a)^{2-\alpha}}+\frac{{}_x^{CPC}D_b^\alpha f(b)}{(b-x)^{2-\alpha}}\right)\right|\\&\leq K_1(\alpha)\left[\frac{1}{p^2(1-\alpha)+p}+\frac{1}{q}\int_0^1\left(t^s\left|f'(a)\right|^q+(1-t)^s\left|f'(x)\right|^q\right)dt\right]\\&+K_0(\alpha)\left[\frac{1}{p^2(1-\alpha)+p}+\frac{1}{q}\int_0^1\left(t^s\left|f''(a)\right|^q+(1-t)^s\left|f''(x)\right|^q\right)dt\right]\\&+K_1(\alpha)\left[\frac{1}{p^2(1-\alpha)+p}+\frac{1}{q}\int_0^1\left(t^s\left|f'(x)\right|^q+(1-t)^s\left|f'(b)\right|^q\right)dt\right]\\&+K_0(\alpha)\left[\frac{1}{p^2(1-\alpha)+p}+\frac{1}{q}\int_0^1\left(t^s\left|f''(x)\right|^q+(1-t)^s\left|f''(b)\right|^q\right)dt\right].\end{aligned}$$

By making necessary computations, we get

$$\begin{aligned}&\left|-\frac{K_1(\alpha)f(a)+K_0(\alpha)f'(a)}{x-a}-\frac{K_1(\alpha)f(x)+K_0(\alpha)f'(x)}{b-x}\right.\\&\left.+\Gamma(2-\alpha)\left(\frac{{}_a^{CPC}D_x^\alpha f(x)}{(x-a)^{2-\alpha}}+\frac{{}_x^{CPC}D_b^\alpha f(b)}{(b-x)^{2-\alpha}}\right)\right|\\&\leq K_1(\alpha)\left[\frac{1}{p^2(1-\alpha)+p}+\frac{\left|f'(a)\right|^q+\left|f'(x)\right|^q}{(s+1)q}\right]\\&+K_0(\alpha)\left[\frac{1}{p^2(1-\alpha)+p}+\frac{\left|f''(a)\right|^q+\left|f''(x)\right|^q}{(s+1)q}\right]\\&+K_1(\alpha)\left[\frac{1}{p^2(1-\alpha)+p}+\frac{\left|f'(x)\right|^q+\left|f'(b)\right|^q}{(s+1)q}\right]\\&+K_0(\alpha)\left[\frac{1}{p^2(1-\alpha)+p}+\frac{\left|f''(x)\right|^q+\left|f''(b)\right|^q}{(s+1)q}\right]\end{aligned}$$

which completes the proof.

Remark 2 *Now, let us briefly consider some special case of the main results. In Theorem 1 Theorem 2, Theorem 3 and Theorem 4, if we choose* $s=1$*, then the main results are reduced to Theorem 1, Theorem 2, Theorem 3 and Theorem 4 by Gürbüz et al. [9].*

Remark 3 *Several special cases can be considered by choosing the functions* $K_0(\alpha)$ *and* $K_1(\alpha)$ *as in Remark 1.*

3. ACKNOWLEDGEMENT

This research is supported by Ordu University Scientific Research Projects Coordination Unit (BAP), Project Number: B-2212.

REFERENCES

1. J.E. Pecaric, D.S. Mitrinovic, A.M. Fink, Classical and New Inequalities in Analysis; Springer: Cham, Switzerland, 1993; pp. 1-2.
2. M.Z. Sarıkaya, E. Set, M.E. Özdemir, S.S. Dragomir, New some Hadamard's type inequalities for co-ordinated convex functions, Tamsui Oxf. J. Math. Sci., 28(2) (2012), 137-152.
3. E. Set, M.E. Özdemir, S.S. Dragomir, On the Hermite-Hadamard inequality and other integral inequalities involving two functions, J. Inequal. Appl., 2010, 2010, 148102.
4. M.Z. Sarıkaya, E. Set, H. Yaldız, N. Başak, Hermite-Hadamards inequalities for fractional integrals and related fractional inequalities, Math. Comput. Model., 57 (2013), 2403-2407.
5. A. Ekinci, M.E. Özdemir, Some New Integral Inequalities via Riemann Liouville Integral Operators, Appl. Comput. Math., 3 (2019), 288-295.
6. M.Z. Sarıkaya, H. Yıldırım, On Hermite-Hadamard type inequalities for Riemann Liouville fractional integrals, Miskolc Math. Notes, 17 (2016), 1049-1059.
7. D. Baleanu, A. Fernandez, A. Akgul, On a fractional operator combining proportional and classical differintegrals, Mathematics, 2020, 8, 360.
8. D.R. Anderson, D.J. Ulness, Newly defined conformable derivatives, Adv. Dyn. Syst. Appl., 10 (2015), 109-137.
9. M. Gürbüz, A.O. Akdemir, M.A. Dokuyucu, Novel Approaches for Differentiable Convex Functions via the Proportional Caputo-Hybrid Operators, Fractal Fract., 2022, 6, 258.

Some Fractional Integral Inequalities Obtained with the Help of Proportional Caputo-Hybrid Operator

Coşkun YALÇINKAYA[1] and Mustafa GÜRBÜZ[2]

[1]Institute of Graduate Education, Ağrı İbrahim Çeçen University, Ağrı-TURKEY
[2]Department of Elementary Mathematics Education, Faculty of Education, Ağrı İbrahim Çeçen University, Ağrı-TURKEY

csknylcnky@gmail.com
mgurbuz@agri.edu.tr

ABSTRACT

In the paper, we establish some new inequalities for differentiable convex functions, which are connected with Hermite-Hadamard-Fejer integral inequalities, and we present new generalized inequalities of trapezoidal type which cover the previously puplished results.

1. INTRODUCTION

Fractional calculus has been appealing to many researchers over the last decades ([4,7]). Some researchers have found that different fractional derivatives with different singular or nonsingular kernels need to be identified by real-world problems in different fields of engineering and science ([8,9]). These different fractional operators are also used in integral inequalities ([1,5,6]). Thus, fractional calculus plays an important role in the development of inequality theory. One of the fractional operators obtained in the last years is so-called Caputo-hybrid operator is given in the following:

Definition 1 *(see [2]) Let $f: I \subseteq \mathbb{R}^+ \to \mathbb{R}$ be a differentiable function on I°. Also let f and f' are L^1 functions on I. Then, the proportional Caputo-hybrid operator may be defined as*

$$ {}_0^{CPC}D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \left(K_1(\alpha) f(x) + K_0(\alpha) f'(x)\right)(t-x)^{-\alpha}\, dx $$

where $\alpha \in [0,1]$ and K_0 and K_1 are functions satisfing

$$ \lim_{\alpha \to 0^+} K_0(\alpha) = 0; \qquad \lim_{\alpha \to 1^-} K_0(\alpha) = 1; \qquad K_0(\alpha) \neq 0, \quad \alpha \in (0,1]; \tag{1.1} $$

$$ \lim_{\alpha \to 0^+} K_1(\alpha) = 1; \qquad \lim_{\alpha \to 1^-} K_1(\alpha) = 0; \qquad K_1(\alpha) \neq 0, \quad \alpha \in [0,1). \tag{1.2} $$

Gürbüz (et. al) obtained the following lemma which we will use to prove some of our results:

Lemma 1 *(see [5]) Let $f: I \subseteq \mathbb{R}$ be a twice differentiable function on I°. Also let f and f' are L^1 functions on I. Then the following equality holds:*

$$
\begin{aligned}
&K_1(\alpha)\int_0^1 t^{1-\alpha} f'(ta+(1-t)x)dt + K_0(\alpha)\int_0^1 t^{1-\alpha} f''(ta+(1-t)x)dt \\
&+K_1(\alpha)\int_0^1 t^{1-\alpha} f'(tx+(1-t)b)dt + K_0(\alpha)\int_0^1 t^{1-\alpha} f''(tx+(1-t)b)dt \\
&= -\frac{K_1(\alpha)f(a)+K_0(\alpha)f'(a)}{x-a} - \frac{K_1(\alpha)f(x)+K_0(\alpha)f'(x)}{b-x} \\
&+\Gamma(2-\alpha)\left(\frac{{}_a^{CPC}D_x^{\alpha} f(x)}{(x-a)^{2-\alpha}} + \frac{{}_x^{CPC}D_b^{\alpha} f(b)}{(b-x)^{2-\alpha}}\right)
\end{aligned}
$$

where $\alpha \in [0,1]$, $a < x < b$ and K_0 and K_1 are functions satisfying (1.1) and (1.2).

In this paper some new inequalities are obtained by using the proportional Caputo-hybrid operator and the lemma given above.

2. SOME RESULTS OBTAINED BY USING A KERNEL

Theorem 1 *Let $f : I \subseteq \mathbb{R}$ be a three times differentiable function on I°. Also let f, f', f'' and f''' are L^1 functions on I. If $|f''|$ and $|f'''|$ are convex on I, then the following inequality holds:*

$$
\begin{aligned}
&(2-\alpha)\left| -\frac{K_1(\alpha)f(a)+K_0(\alpha)f'(a)}{x-a} - \frac{K_1(\alpha)f(x)+K_0(\alpha)f'(x)}{b-x} \right. \\
&\left. +\Gamma(2-\alpha)\left(\frac{{}_a^{CPC}D_x^{\alpha} f(x)}{(x-a)^{2-\alpha}} + \frac{{}_x^{CPC}D_b^{\alpha} f(b)}{(b-x)^{2-\alpha}}\right)\right| \\
&\le K_1(\alpha)\left(|f'(a)| + (x-a)\left(\frac{|f''(a)|}{4-\alpha} + \frac{|f''(x)|}{(3-\alpha)(4-\alpha)}\right)\right) \\
&+K_0(\alpha)\left(|f''(a)| + (x-a)\left(\frac{|f'''(a)|}{4-\alpha} + \frac{|f'''(x)|}{(3-\alpha)(4-\alpha)}\right)\right) \\
&+K_1(\alpha)\left(|f'(x)| + (b-x)\left(\frac{|f''(x)|}{4-\alpha} + \frac{|f''(b)|}{(3-\alpha)(4-\alpha)}\right)\right) \\
&+K_0(\alpha)\left(|f''(x)| + (b-x)\left(\frac{|f'''(x)|}{4-\alpha} + \frac{|f'''(b)|}{(3-\alpha)(4-\alpha)}\right)\right)
\end{aligned}
$$

where $\alpha \in [0,1]$, $a, b \in I$, $a < x < b$, K_0 and K_1 are functions satisfying (1.1) and (1.2).

Proof. Using Lemma 1 and integrating by parts we get

$$
\begin{aligned}
&-\frac{K_1(\alpha)f(a)+K_0(\alpha)f'(a)}{x-a}-\frac{K_1(\alpha)f(x)+K_0(\alpha)f'(x)}{b-x}\\
&+\Gamma(2-\alpha)\left(\frac{{}_{a}^{CPC}D_x^{\alpha}f(x)}{(x-a)^{2-\alpha}}+\frac{{}_{x}^{CPC}D_b^{\alpha}f(b)}{(b-x)^{2-\alpha}}\right)\\
&=K_1(\alpha)\left(\frac{t^{2-\alpha}}{2-\alpha}f'(ta+(1-t)x)\Big|_{t=0}^{t=1}-\frac{(a-x)}{2-\alpha}\int_0^1 t^{2-\alpha}f''(ta+(1-t)x)dt\right)\\
&+K_0(\alpha)\left(\frac{t^{2-\alpha}}{2-\alpha}f''(ta+(1-t)x)\Big|_{t=0}^{t=1}-\frac{(a-x)}{2-\alpha}\int_0^1 t^{2-\alpha}f'''(ta+(1-t)x)dt\right)\\
&+K_1(\alpha)\left(\frac{t^{2-\alpha}}{2-\alpha}f'(tx+(1-t)b)\Big|_{t=0}^{t=1}-\frac{(x-b)}{2-\alpha}\int_0^1 t^{2-\alpha}f''(tx+(1-t)b)dt\right)\\
&+K_0(\alpha)\left(\frac{t^{2-\alpha}}{2-\alpha}f''(tx+(1-t)b)\Big|_{t=0}^{t=1}-\frac{(x-b)}{2-\alpha}\int_0^1 t^{2-\alpha}f'''(tx+(1-t)b)dt\right)
\end{aligned}
$$

which yields

$$
\begin{aligned}
&-\frac{K_1(\alpha)f(a)+K_0(\alpha)f'(a)}{x-a}-\frac{K_1(\alpha)f(x)+K_0(\alpha)f'(x)}{b-x}\\
&+\Gamma(2-\alpha)\left(\frac{{}_{a}^{CPC}D_x^{\alpha}f(x)}{(x-a)^{2-\alpha}}+\frac{{}_{x}^{CPC}D_b^{\alpha}f(b)}{(b-x)^{2-\alpha}}\right)\\
&=K_1(\alpha)\left(\frac{f'(a)}{2-\alpha}+\frac{x-a}{2-\alpha}\int_0^1 t^{2-\alpha}f''(ta+(1-t)x)dt\right)\\
&+K_0(\alpha)\left(\frac{f''(a)}{2-\alpha}+\frac{x-a}{2-\alpha}\int_0^1 t^{2-\alpha}f'''(ta+(1-t)x)dt\right)\\
&+K_1(\alpha)\left(\frac{f'(x)}{2-\alpha}+\frac{b-x}{2-\alpha}\int_0^1 t^{2-\alpha}f''(tx+(1-t)b)dt\right)\\
&+K_0(\alpha)\left(\frac{f''(x)}{2-\alpha}+\frac{b-x}{2-\alpha}\int_0^1 t^{2-\alpha}f'''(tx+(1-t)b)dt\right).
\end{aligned}
$$

With the help of properties of modulus we have

$$
\begin{aligned}
&\Bigg|-\frac{K_1(\alpha)f(a)+K_0(\alpha)f'(a)}{x-a}-\frac{K_1(\alpha)f(x)+K_0(\alpha)f'(x)}{b-x}\\
&+\Gamma(2-\alpha)\left(\frac{{}_{a}^{CPC}D_x^{\alpha}f(x)}{(x-a)^{2-\alpha}}+\frac{{}_{x}^{CPC}D_b^{\alpha}f(b)}{(b-x)^{2-\alpha}}\right)\Bigg|
\end{aligned}
$$

$$\leq K_1(\alpha)\left(\frac{|f'(a)|}{2-\alpha}+\frac{x-a}{2-\alpha}\int_0^1 t^{2-\alpha}|f''(ta+(1-t)x)|dt\right)$$
$$+K_0(\alpha)\left(\frac{|f''(a)|}{2-\alpha}+\frac{x-a}{2-\alpha}\int_0^1 t^{2-\alpha}|f'''(ta+(1-t)x)|dt\right)$$
$$+K_1(\alpha)\left(\frac{|f'(x)|}{2-\alpha}+\frac{b-x}{2-\alpha}\int_0^1 t^{2-\alpha}|f''(tx+(1-t)b)|dt\right)$$
$$+K_0(\alpha)\left(\frac{|f''(x)|}{2-\alpha}+\frac{b-x}{2-\alpha}\int_0^1 t^{2-\alpha}|f'''(tx+(1-t)b)|dt\right) \tag{2.1}$$

By using convexity of $|f''|$ and $|f'''|$ we get

$$\left|-\frac{K_1(\alpha)f(a)+K_0(\alpha)f'(a)}{x-a}-\frac{K_1(\alpha)f(x)+K_0(\alpha)f'(x)}{b-x}\right.$$
$$\left.+\Gamma(2-\alpha)\left(\frac{{}_a^{CPC}D_x^{\alpha}f(x)}{(x-a)^{2-\alpha}}+\frac{{}_x^{CPC}D_b^{\alpha}f(b)}{(b-x)^{2-\alpha}}\right)\right|$$
$$\leq K_1(\alpha)\left(\frac{|f'(a)|}{2-\alpha}+\frac{x-a}{2-\alpha}\int_0^1 t^{2-\alpha}\left(t|f''(a)|+(1-t)|f''(x)|\right)dt\right)$$
$$+K_0(\alpha)\left(\frac{|f''(a)|}{2-\alpha}+\frac{x-a}{2-\alpha}\int_0^1 t^{2-\alpha}\left(t|f'''(a)|+(1-t)|f'''(x)|\right)dt\right)$$
$$+K_1(\alpha)\left(\frac{|f'(x)|}{2-\alpha}+\frac{b-x}{2-\alpha}\int_0^1 t^{2-\alpha}\left(t|f''(x)|+(1-t)|f''(b)|\right)dt\right)$$
$$+K_0(\alpha)\left(\frac{|f''(x)|}{2-\alpha}+\frac{b-x}{2-\alpha}\int_0^1 t^{2-\alpha}\left(t|f'''(x)|+(1-t)|f'''(b)|\right)dt\right).$$

With simple calculations we get

$$\left|-\frac{K_1(\alpha)f(a)+K_0(\alpha)f'(a)}{x-a}-\frac{K_1(\alpha)f(x)+K_0(\alpha)f'(x)}{b-x}\right.$$
$$\left.+\Gamma(2-\alpha)\left(\frac{{}_a^{CPC}D_x^{\alpha}f(x)}{(x-a)^{2-\alpha}}+\frac{{}_x^{CPC}D_b^{\alpha}f(b)}{(b-x)^{2-\alpha}}\right)\right|$$
$$\leq K_1(\alpha)\left(\frac{|f'(a)|}{2-\alpha}+\frac{x-a}{2-\alpha}\left(\frac{|f''(a)|}{4-\alpha}+\frac{|f''(x)|}{(3-\alpha)(4-\alpha)}\right)\right)$$

$$+K_0(\alpha)\left(\frac{|f''(a)|}{2-\alpha}+\frac{x-a}{2-\alpha}\left(\frac{|f'''(a)|}{4-\alpha}+\frac{|f'''(x)|}{(3-\alpha)(4-\alpha)}\right)\right)$$
$$+K_1(\alpha)\left(\frac{|f'(x)|}{2-\alpha}+\frac{b-x}{2-\alpha}\left(\frac{|f''(x)|}{4-\alpha}+\frac{|f''(b)|}{(3-\alpha)(4-\alpha)}\right)\right)$$
$$+K_0(\alpha)\left(\frac{|f''(x)|}{2-\alpha}+\frac{b-x}{2-\alpha}\left(\frac{|f'''(x)|}{4-\alpha}+\frac{|f'''(b)|}{(3-\alpha)(4-\alpha)}\right)\right)$$

which completes the proof.

Corollary 1 *Under the conditions of Theorem 1, if we choose* $\alpha=\frac{1}{2}$ *then we get*

$$\frac{3}{2}\left|-\frac{K_1\left(\frac{1}{2}\right)f(a)+K_0\left(\frac{1}{2}\right)f'(a)}{x-a}-\frac{K_1\left(\frac{1}{2}\right)f(x)+K_0\left(\frac{1}{2}\right)f'(x)}{b-x}\right.$$
$$\left.+\frac{\sqrt{\pi}}{2}\left(\frac{{}^{CPC}_{a}D_x^{\frac{1}{2}}f(x)}{(x-a)^{\frac{3}{2}}}+\frac{{}^{CPC}_{x}D_b^{\frac{1}{2}}f(b)}{(b-x)^{\frac{3}{2}}}\right)\right|$$
$$\le K_1\left(\frac{1}{2}\right)\left(|f'(a)|+(x-a)\left(\frac{10|f''(a)|+4|f''(x)|}{35}\right)\right)$$
$$+K_0\left(\frac{1}{2}\right)\left(|f''(a)|+(x-a)\left(\frac{10|f'''(a)|+4|f'''(x)|}{35}\right)\right)$$
$$+K_1\left(\frac{1}{2}\right)\left(|f'(x)|+(b-x)\left(\frac{10|f''(x)|+4|f''(b)|}{35}\right)\right)$$
$$+K_0\left(\frac{1}{2}\right)\left(|f''(x)|+(b-x)\left(\frac{10|f'''(x)|+4|f'''(b)|}{35}\right)\right).$$

Theorem 2 *Let* $f:I\subseteq\mathbb{R}$ *be a three times differentiable function on* I°. *Also let* f, f', f'' *and* f''' *are* L^1 *functions on* I. *If* $|f''|^q$ *and* $|f'''|^q$ *are convex on* I, *then the following inequality holds:*

$$
\begin{aligned}
&(2-\alpha)\left|-\frac{K_1(\alpha)f(a)+K_0(\alpha)f'(a)}{x-a}-\frac{K_1(\alpha)f(x)+K_0(\alpha)f'(x)}{b-x}\right.\\
&\left.+\Gamma(2-\alpha)\left(\frac{{}^{CPC}_{a}D^{\alpha}_{x}f(x)}{(x-a)^{2-\alpha}}+\frac{{}^{CPC}_{x}D^{\alpha}_{b}f(b)}{(b-x)^{2-\alpha}}\right)\right|\\
&\leq K_1(\alpha)\left(\left|f'(a)\right|+C(x-a)\left(\frac{\left|f''(a)\right|^q+\left|f''(x)\right|^q}{2}\right)^{\frac{1}{q}}\right)\\
&+K_0(\alpha)\left(\left|f''(a)\right|+C(x-a)\left(\frac{\left|f'''(a)\right|^q+\left|f'''(x)\right|^q}{2}\right)^{\frac{1}{q}}\right)\\
&+K_1(\alpha)\left(\left|f'(x)\right|+C(b-x)\left(\frac{\left|f''(x)\right|^q+\left|f''(b)\right|^q}{2}\right)^{\frac{1}{q}}\right)\\
&+K_0(\alpha)\left(\left|f''(x)\right|+C(b-x)\left(\frac{\left|f'''(x)\right|^q+\left|f'''(b)\right|^q}{2}\right)^{\frac{1}{q}}\right)
\end{aligned}
$$

where $C=\left(\frac{1}{1+p(2-\alpha)}\right)^{\frac{1}{p}}$, $\alpha\in[0,1]$, $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $a,b\in I$, $a<x<b$, K_0 and K_1 are functions satisfying (1.1) and (1.2).

Proof. Using (2.1) and Hölder's inequality we get

$$
\begin{aligned}
&\left|-\frac{K_1(\alpha)f(a)+K_0(\alpha)f'(a)}{x-a}-\frac{K_1(\alpha)f(x)+K_0(\alpha)f'(x)}{b-x}\right.\\
&\left.+\Gamma(2-\alpha)\left(\frac{{}^{CPC}_{a}D^{\alpha}_{x}f(x)}{(x-a)^{2-\alpha}}+\frac{{}^{CPC}_{x}D^{\alpha}_{b}f(b)}{(b-x)^{2-\alpha}}\right)\right|\\
&\leq K_1(\alpha)\left(\frac{\left|f'(a)\right|}{2-\alpha}+\frac{x-a}{2-\alpha}\left(\int_0^1 t^{(2-\alpha)p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f''(ta+(1-t)x)\right|^q dt\right)^{\frac{1}{q}}\right)\\
&+K_0(\alpha)\left(\frac{\left|f''(a)\right|}{2-\alpha}+\frac{x-a}{2-\alpha}\left(\int_0^1 t^{(2-\alpha)p}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'''(ta+(1-t)x)\right|^q dt\right)^{\frac{1}{q}}\right)
\end{aligned}
$$

$$+K_1(\alpha)\left(\frac{|f'(x)|}{2-\alpha}+\frac{b-x}{2-\alpha}\left(\int_0^1 t^{(2-\alpha)p}dt\right)^{\frac{1}{p}}\left(\int_0^1 |f''(tx+(1-t)b)|^q dt\right)^{\frac{1}{q}}\right)$$

$$+K_0(\alpha)\left(\frac{|f''(x)|}{2-\alpha}+\frac{b-x}{2-\alpha}\left(\int_0^1 t^{(2-\alpha)p}dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'''(tx+(1-t)b)|^q dt\right)^{\frac{1}{q}}\right)$$

Using convexity of $|f''|^q$ and $|f'''|^q$ we have

$$\left|-\frac{K_1(\alpha)f(a)+K_0(\alpha)f'(a)}{x-a}-\frac{K_1(\alpha)f(x)+K_0(\alpha)f'(x)}{b-x}\right.$$
$$\left.+\Gamma(2-\alpha)\left(\frac{{}^{CPC}_{\ a}D_x^{\alpha} f(x)}{(x-a)^{2-\alpha}}+\frac{{}^{CPC}_{\ x}D_b^{\alpha} f(b)}{(b-x)^{2-\alpha}}\right)\right|$$
$$\leq K_1(\alpha)\left(\frac{|f'(a)|}{2-\alpha}+\frac{x-a}{2-\alpha}\left(\frac{1}{1+p(2-\alpha)}\right)^{\frac{1}{p}}\left(\int_0^1\left(t|f''(a)|^q+(1-t)|f''(x)|^q\right)dt\right)^{\frac{1}{q}}\right)$$
$$+K_0(\alpha)\left(\frac{|f''(a)|}{2-\alpha}+\frac{x-a}{2-\alpha}\left(\frac{1}{1+p(2-\alpha)}\right)^{\frac{1}{p}}\left(\int_0^1\left(t|f'''(a)|^q+(1-t)|f'''(x)|^q\right)dt\right)^{\frac{1}{q}}\right)$$
$$+K_1(\alpha)\left(\frac{|f'(x)|}{2-\alpha}+\frac{b-x}{2-\alpha}\left(\frac{1}{1+p(2-\alpha)}\right)^{\frac{1}{p}}\left(\int_0^1\left(t|f''(x)|^q+(1-t)|f''(b)|^q\right)dt\right)^{\frac{1}{q}}\right)$$
$$+K_0(\alpha)\left(\frac{|f''(x)|}{2-\alpha}+\frac{b-x}{2-\alpha}\left(\frac{1}{1+p(2-\alpha)}\right)^{\frac{1}{p}}\left(\int_0^1\left(t|f'''(x)|^q+(1-t)|f'''(b)|^q\right)dt\right)^{\frac{1}{q}}\right)$$

By making necessary computations, we get

$$\left|-\frac{K_1(\alpha)f(a)+K_0(\alpha)f'(a)}{x-a}-\frac{K_1(\alpha)f(x)+K_0(\alpha)f'(x)}{b-x}\right.$$
$$\left.+\Gamma(2-\alpha)\left(\frac{{}^{CPC}_{\ a}D_x^{\alpha} f(x)}{(x-a)^{2-\alpha}}+\frac{{}^{CPC}_{\ x}D_b^{\alpha} f(b)}{(b-x)^{2-\alpha}}\right)\right|$$
$$\leq K_1(\alpha)\left(\frac{|f'(a)|}{2-\alpha}+\frac{x-a}{2-\alpha}\left(\frac{1}{1+p(2-\alpha)}\right)^{\frac{1}{p}}\left(\frac{|f''(a)|^q+|f''(x)|^q}{2}\right)^{\frac{1}{q}}\right)$$

$$+K_0(\alpha)\left(\frac{|f''(a)|}{2-\alpha}+\frac{x-a}{2-\alpha}\left(\frac{1}{1+p(2-\alpha)}\right)^{\frac{1}{p}}\left(\frac{|f'''(a)|^q+|f'''(x)|^q}{2}\right)^{\frac{1}{q}}\right)$$

$$+K_1(\alpha)\left(\frac{|f'(x)|}{2-\alpha}+\frac{b-x}{2-\alpha}\left(\frac{1}{1+p(2-\alpha)}\right)^{\frac{1}{p}}\left(\frac{|f''(x)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}}\right)$$

$$+K_0(\alpha)\left(\frac{|f''(x)|}{2-\alpha}+\frac{b-x}{2-\alpha}\left(\frac{1}{1+p(2-\alpha)}\right)^{\frac{1}{p}}\left(\frac{|f'''(x)|^q+|f'''(b)|^q}{2}\right)^{\frac{1}{q}}\right)$$

which completes proof.

Theorem 3 *Let $f: I \subseteq \mathbb{R}$ be a three times differentiable function on I°. Also let f, f', f'' and f''' are L^1 functions on I. If $|f''|^q$ and $|f'''|^q$ are convex on I, then the following inequality holds:*

$$(2-\alpha)\left|-\frac{K_1(\alpha)f(a)+K_0(\alpha)f'(a)}{x-a}-\frac{K_1(\alpha)f(x)+K_0(\alpha)f'(x)}{b-x}\right.$$

$$\left.+\Gamma(2-\alpha)\left(\frac{{}_{a}^{CPC}D_x^\alpha f(x)}{(x-a)^{2-\alpha}}+\frac{{}_{x}^{CPC}D_b^\alpha f(b)}{(b-x)^{2-\alpha}}\right)\right|$$

$$\leq K_1(\alpha)\left(|f'(a)|+(x-a)\left[D+\frac{|f''(a)|^q+|f''(x)|^q}{2q}\right]\right)$$

$$+K_0(\alpha)\left(|f''(a)|+(x-a)\left[D+\frac{|f'''(a)|^q+|f'''(x)|^q}{2q}\right]\right)$$

$$+K_1(\alpha)\left(|f'(x)|+(b-x)\left[D+\frac{|f''(x)|^q+|f''(b)|^q}{2q}\right]\right)$$

$$+K_0(\alpha)\left(|f''(x)|+(b-x)\left[D+\frac{|f'''(x)|^q+|f'''(b)|^q}{2q}\right]\right)$$

where $D=\dfrac{1}{p+p^2(2-\alpha)}$, $\alpha\in[0,1]$, $p>1$, $\dfrac{1}{p}+\dfrac{1}{q}=1$, $a,b\in I$, $a<x<b$, K_0 and K_1 are functions satisfying (1.1) and (1.2).

Proof. Using (2.1) and Young's inequality we get

$$\left| -\frac{K_1(\alpha)f(a)+K_0(\alpha)f'(a)}{x-a} - \frac{K_1(\alpha)f(x)+K_0(\alpha)f'(x)}{b-x} \right.$$
$$\left. +\Gamma(2-\alpha)\left(\frac{{}^{CPC}_{a}D^{\alpha}_{x} f(x)}{(x-a)^{2-\alpha}} + \frac{{}^{CPC}_{x}D^{\alpha}_{b} f(b)}{(b-x)^{2-\alpha}} \right) \right|$$
$$\le K_1(\alpha)\left(\frac{|f'(a)|}{2-\alpha} + \frac{x-a}{2-\alpha}\left[\frac{1}{p}\int_0^1 t^{(2-\alpha)p}dt + \frac{1}{q}\int_0^1 \left|f''(ta+(1-t)x)\right|^q dt \right] \right)$$
$$+K_0(\alpha)\left(\frac{|f''(a)|}{2-\alpha} + \frac{x-a}{2-\alpha}\left[\frac{1}{p}\int_0^1 t^{(2-\alpha)p}dt + \frac{1}{q}\int_0^1 \left|f'''(ta+(1-t)x)\right|^q dt \right] \right)$$
$$+K_1(\alpha)\left(\frac{|f'(x)|}{2-\alpha} + \frac{b-x}{2-\alpha}\left[\frac{1}{p}\int_0^1 t^{(2-\alpha)p}dt + \frac{1}{q}\int_0^1 \left|f''(tx+(1-t)b)\right|^q dt \right] \right)$$
$$+K_0(\alpha)\left(\frac{|f''(x)|}{2-\alpha} + \frac{b-x}{2-\alpha}\left[\frac{1}{p}\int_0^1 t^{(2-\alpha)p}dt + \frac{1}{q}\int_0^1 \left|f'''(tx+(1-t)b)\right|^q dt \right] \right)$$

Using convexity of $\left|f''\right|^q$ and $\left|f'''\right|^q$ we have

$$\left| -\frac{K_1(\alpha)f(a)+K_0(\alpha)f'(a)}{x-a} - \frac{K_1(\alpha)f(x)+K_0(\alpha)f'(x)}{b-x} \right.$$
$$\left. +\Gamma(2-\alpha)\left(\frac{{}^{CPC}_{a}D^{\alpha}_{x} f(x)}{(x-a)^{2-\alpha}} + \frac{{}^{CPC}_{x}D^{\alpha}_{b} f(b)}{(b-x)^{2-\alpha}} \right) \right|$$
$$\le K_1(\alpha)\left(\frac{|f'(a)|}{2-\alpha} + \frac{x-a}{2-\alpha}\left[\frac{1}{p+p^2(2-\alpha)} + \frac{1}{q}\int_0^1 \left(t\left|f''(a)\right|^q + (1-t)\left|f''(x)\right|^q \right)dt \right] \right)$$
$$+K_0(\alpha)\left(\frac{|f''(a)|}{2-\alpha} + \frac{x-a}{2-\alpha}\left[\frac{1}{p+p^2(2-\alpha)} + \frac{1}{q}\int_0^1 \left(t\left|f'''(a)\right|^q + (1-t)\left|f'''(x)\right|^q \right)dt \right] \right)$$
$$+K_1(\alpha)\left(\frac{|f'(x)|}{2-\alpha} + \frac{b-x}{2-\alpha}\left[\frac{1}{p+p^2(2-\alpha)} + \frac{1}{q}\int_0^1 \left(t\left|f''(x)\right|^q + (1-t)\left|f''(b)\right|^q \right)dt \right] \right)$$
$$+K_0(\alpha)\left(\frac{|f''(x)|}{2-\alpha} + \frac{b-x}{2-\alpha}\left[\frac{1}{p+p^2(2-\alpha)} + \frac{1}{q}\int_0^1 \left(t\left|f'''(x)\right|^q + (1-t)\left|f'''(b)\right|^q \right)dt \right] \right)$$

By making necessary computations, we get

$$
\begin{aligned}
&\Bigg| -\frac{K_1(\alpha)f(a)+K_0(\alpha)f'(a)}{x-a}-\frac{K_1(\alpha)f(x)+K_0(\alpha)f'(x)}{b-x} \\
&+\Gamma(2-\alpha)\left(\frac{{}_{a}^{CPC}D_x^{\alpha}f(x)}{(x-a)^{2-\alpha}}+\frac{{}_{x}^{CPC}D_b^{\alpha}f(b)}{(b-x)^{2-\alpha}}\right)\Bigg| \\
&\leq K_1(\alpha)\left(\frac{|f'(a)|}{2-\alpha}+\frac{x-a}{2-\alpha}\left[\frac{1}{p+p^2(2-\alpha)}+\frac{|f''(a)|^q+|f''(x)|^q}{2q}\right]\right) \\
&+K_0(\alpha)\left(\frac{|f''(a)|}{2-\alpha}+\frac{x-a}{2-\alpha}\left[\frac{1}{p+p^2(2-\alpha)}+\frac{|f'''(a)|^q+|f'''(x)|^q}{2q}\right]\right) \\
&+K_1(\alpha)\left(\frac{|f'(x)|}{2-\alpha}+\frac{b-x}{2-\alpha}\left[\frac{1}{p+p^2(2-\alpha)}+\frac{|f''(x)|^q+|f''(b)|^q}{2q}\right]\right) \\
&+K_0(\alpha)\left(\frac{|f''(x)|}{2-\alpha}+\frac{b-x}{2-\alpha}\left[\frac{1}{p+p^2(2-\alpha)}+\frac{|f'''(x)|^q+|f'''(b)|^q}{2q}\right]\right)
\end{aligned}
$$

which completes the proof.

3. SOME RESULTS OBTAINED FOR BOUNDED FUNCTIONS

Theorem 4 *Let* $f: I \subseteq \mathbb{R}^+ \to \mathbb{R}$ *be a differentiable function on* I°. *Also let* f, f', g_1 *and* g_2 *are* L^1 *functions on* I *such that*

$$
\begin{aligned}
g_1(t) &\leq f(t) \leq g_2(t) \\
g_1(t) &\leq f'(t) \leq g_2(t)
\end{aligned}
\tag{3.1}
$$

Then, the following inequality holds:

$$
\begin{aligned}
&\left({}_{0}^{CPC}D_u^{\alpha}f(u)\right)\left(\int_0^u g_1(x)(u-x)^{-\alpha}dx+\int_0^u g_2(x)(u-x)^{-\alpha}dx\right) \\
&\geq \frac{(K_0(\alpha)+K_1(\alpha))}{\Gamma(1-\alpha)}\int_0^u g_1(x)(u-x)^{-\alpha}dx\int_0^u g_2(x)(u-x)^{-\alpha}dx \\
&+\frac{K_1(\alpha)}{\Gamma(1-\alpha)}\left(\int_0^u f(x)(u-x)^{-\alpha}dx\right)^2 \\
&+\frac{K_0(\alpha)}{\Gamma(1-\alpha)}\left(\int_0^u f'(x)(u-x)^{-\alpha}dx\right)^2.
\end{aligned}
\tag{3.2}
$$

where $\alpha \in [0,1]$ with K_0 and K_1 are functions satisfing the conditions (1.1) and (1.2).

Proof. From (3.1) for all $x \geq 0$, $y \geq 0$ we have

$$\begin{aligned}&(g_2(x)-f(x))(f(y)-g_1(y))\geq 0\\&g_2(x)f(y)+f(x)g_1(y)\geq g_1(y)g_2(x)+f(x)f(y)\end{aligned}\tag{3.3}$$

and

$$\begin{aligned}&(g_2(x)-f'(x))(f'(y)-g_1(y))\geq 0\\&g_2(x)f'(y)+f'(x)g_1(y)\geq g_1(y)g_2(x)+f'(x)f'(y).\end{aligned}\tag{3.4}$$

Multiplying (3.3) by $\frac{K_1(\alpha)(u-x)^{-\alpha}}{\Gamma(1-\alpha)}$ and (3.4) by $\frac{K_0(\alpha)(u-x)^{-\alpha}}{\Gamma(1-\alpha)}$ then adding the resulting inequalities we get

$$\begin{aligned}&\frac{K_1(\alpha)(u-x)^{-\alpha}}{\Gamma(1-\alpha)}[g_2(x)f(y)+f(x)g_1(y)]\\&+\frac{K_0(\alpha)(u-x)^{-\alpha}}{\Gamma(1-\alpha)}[g_2(x)f'(y)+f'(x)g_1(y)]\\&\geq\frac{K_1(\alpha)(u-x)^{-\alpha}}{\Gamma(1-\alpha)}[g_1(y)g_2(x)+f(x)f(y)]\\&+\frac{K_0(\alpha)(u-x)^{-\alpha}}{\Gamma(1-\alpha)}[g_1(y)g_2(x)+f'(x)f'(y)]\end{aligned}\tag{3.5}$$

which yields

$$\begin{aligned}&\frac{g_2(x)(u-x)^{-\alpha}}{\Gamma(1-\alpha)}[K_1(\alpha)f(y)+K_0(\alpha)f'(y)]\\&+\frac{g_1(y)(u-x)^{-\alpha}}{\Gamma(1-\alpha)}[K_1(\alpha)f(x)+K_0(\alpha)f'(x)]\\&\geq\frac{g_1(y)g_2(x)(u-x)^{-\alpha}}{\Gamma(1-\alpha)}(K_1(\alpha)+K_0(\alpha))\\&+\frac{(u-x)^{-\alpha}}{\Gamma(1-\alpha)}(K_1(\alpha)f(x)f(y)+K_0(\alpha)f'(x)f'(y)).\end{aligned}\tag{3.6}$$

Integrating with respect to x from 0 to u

$$\begin{aligned}&\frac{[K_1(\alpha)f(y)+K_0(\alpha)f'(y)]}{\Gamma(1-\alpha)}\int_0^u g_2(x)(u-x)^{-\alpha}dx\\&+g_1(y)\left({}_0^{CPC}D_u^{\alpha}f(u)\right)\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{g_1(y)(K_1(\alpha)+K_0(\alpha))}{\Gamma(1-\alpha)}\int_0^u g_2(x)(u-x)^{-\alpha}\,dx \\
&+\frac{K_1(\alpha)f(y)}{\Gamma(1-\alpha)}\int_0^u f(x)(u-x)^{-\alpha}\,dx \\
&+\frac{K_0(\alpha)f'(y)}{\Gamma(1-\alpha)}\int_0^u f'(x)(u-x)^{-\alpha}\,dx.
\end{aligned}
\tag{3.7}
$$

Multiplying (3.7) by $(u-y)^{-\alpha}$ and integrating with respect to y from 0 to u

$$
\begin{aligned}
&\left({}_0^{CPC}D_u^{\alpha} f(u)\right)\int_0^u g_2(x)(u-x)^{-\alpha}\,dx \\
&+\left({}_0^{CPC}D_u^{\alpha} f(u)\right)\int_0^u g_1(y)(u-y)^{-\alpha}\,dy \\
&\geq \frac{(K_1(\alpha)+K_0(\alpha))}{\Gamma(1-\alpha)}\int_0^u g_2(x)(u-x)^{-\alpha}\,dx\int_0^u g_1(y)(u-y)^{-\alpha}\,dy \\
&+\frac{K_1(\alpha)}{\Gamma(1-\alpha)}\int_0^u f(x)(u-x)^{-\alpha}\,dx\int_0^u f(y)(u-y)^{-\alpha}\,dy \\
&+\frac{K_0(\alpha)}{\Gamma(1-\alpha)}\int_0^u f'(x)(u-x)^{-\alpha}\,dx\int_0^u f'(y)(u-y)^{-\alpha}\,dy.
\end{aligned}
\tag{3.8}
$$

By rearranging (3.8) we get

$$
\begin{aligned}
&\left({}_0^{CPC}D_u^{\alpha} f(u)\right)\left(\int_0^u g_1(x)(u-x)^{-\alpha}\,dx+\int_0^u g_2(x)(u-x)^{-\alpha}\,dx\right) \\
&\geq \frac{(K_1(\alpha)+K_0(\alpha))}{\Gamma(1-\alpha)}\int_0^u g_1(x)(u-x)^{-\alpha}\,dx\int_0^u g_2(x)(u-x)^{-\alpha}\,dx \\
&+\frac{K_1(\alpha)}{\Gamma(1-\alpha)}\left(\int_0^u f(x)(u-x)^{-\alpha}\,dx\right)^2 \\
&+\frac{K_0(\alpha)}{\Gamma(1-\alpha)}\left(\int_0^u f'(x)(u-x)^{-\alpha}\,dx\right)^2.
\end{aligned}
\tag{3.9}
$$

So the proof is completed.

Corollary 2 *Under the conditions of Theorem 4 if we choose* $\alpha=\frac{1}{2}$, *we get*

$$
\begin{aligned}
&\left({}_0^{CPC}D_u^{\frac{1}{2}} f(u)\right)\left(\int_0^u \frac{g_1(x)}{\sqrt{u-x}}\,dx+\int_0^u \frac{g_2(x)}{\sqrt{u-x}}\,dx\right) \\
&\geq \frac{\left(K_0\left(\frac{1}{2}\right)+K_1\left(\frac{1}{2}\right)\right)}{\sqrt{\pi}}\int_0^u \frac{g_1(x)}{\sqrt{u-x}}\,dx\int_0^u \frac{g_2(x)}{\sqrt{u-x}}\,dx
\end{aligned}
$$

$$+\frac{K_1\left(\frac{1}{2}\right)}{\sqrt{\pi}}\left(\int_0^u \frac{f(x)}{\sqrt{u-x}}dx\right)^2$$
$$+\frac{K_0\left(\frac{1}{2}\right)}{\sqrt{\pi}}\left(\int_0^u \frac{f'(x)}{\sqrt{u-x}}dx\right)^2.$$

Theorem 5 *Let* $f: I \subseteq \mathbb{R}^+ \to \mathbb{R}$ *be a differentiable function on* I°. *Also let* f, f', h, h', g_1, g_2, v_1 *and* v_2 *are* L^1 *functions on* I *satisfying*

$$v_1(t) \le h(t) \le v_2(t)$$
$$v_1(t) \le h'(t) \le v_2(t) \tag{3.10}$$

and condition (3.1). Then, the following inequality holds:

$$\left({}_0^{CPC}D_u^\alpha f(u)\right)\left(\int_0^u g_2(x)(u-x)^{-\alpha}dx + \int_0^u v_1(x)(u-x)^{-\alpha}dx\right)$$
$$\ge \frac{(K_1(\alpha)+K_0(\alpha))}{\Gamma(1-\alpha)}\int_0^u g_2(x)(u-x)^{-\alpha}dx\int_0^u v_1(x)(u-x)^{-\alpha}dx$$
$$+\frac{K_1(\alpha)}{\Gamma(1-\alpha)}\int_0^u f(x)(u-x)^{-\alpha}dx\int_0^u h(x)(u-x)^{-\alpha}dx$$
$$+\frac{K_0(\alpha)}{\Gamma(1-\alpha)}\int_0^u f'(x)(u-x)^{-\alpha}dx\int_0^u h'(x)(u-x)^{-\alpha}dx.$$

where $\alpha \in [0,1]$ with K_0 and K_1 are functions satisfing the conditions (1.1) and (1.2).

Proof. From (3.1) and (3.10) for all $x \ge 0$, $y \ge 0$ we have

$$(g_2(x)-f(x))(h(y)-v_1(y)) \ge 0$$
$$g_2(x)h(y)+v_1(y)f(x) \ge v_1(y)g_2(x)+f(x)h(y) \tag{3.11}$$

and

$$(g_2(x)-f'(x))(h'(y)-v_1(y)) \ge 0$$
$$g_2(x)h'(y)+v_1(y)f'(x) \ge v_1(y)g_2(x)+f'(x)h'(y) \tag{3.12}$$

Multiplying (3.11) by $\frac{K_1(\alpha)(u-x)^{-\alpha}}{\Gamma(1-\alpha)}$ and (3.12) by $\frac{K_0(\alpha)(u-x)^{-\alpha}}{\Gamma(1-\alpha)}$ then adding the resulting inequalities we get

$$
\begin{aligned}
&\frac{K_1(\alpha)(u-x)^{-\alpha}}{\Gamma(1-\alpha)}\left[g_2(x)h(y)+v_1(y)f(x)\right] \\
&+\frac{K_0(\alpha)(u-x)^{-\alpha}}{\Gamma(1-\alpha)}\left[g_2(x)h'(y)+v_1(y)f'(x)\right] \\
&\geq \frac{K_1(\alpha)(u-x)^{-\alpha}}{\Gamma(1-\alpha)}\left[v_1(y)g_2(x)+f(x)h(y)\right] \\
&+\frac{K_0(\alpha)(u-x)^{-\alpha}}{\Gamma(1-\alpha)}\left[v_1(y)g_2(x)+f'(x)h'(y)\right]
\end{aligned}
\tag{3.13}
$$

which yields

$$
\begin{aligned}
&\frac{g_2(x)(u-x)^{-\alpha}}{\Gamma(1-\alpha)}\left[K_1(\alpha)h(y)+K_0(\alpha)h'(y)\right] \\
&+\frac{v_1(y)(u-x)^{-\alpha}}{\Gamma(1-\alpha)}\left[K_1(\alpha)f(x)+K_0(\alpha)f'(x)\right] \\
&\geq \frac{v_1(y)g_2(x)(u-x)^{-\alpha}}{\Gamma(1-\alpha)}\left(K_1(\alpha)+K_0(\alpha)\right) \\
&+\frac{(u-x)^{-\alpha}}{\Gamma(1-\alpha)}\left(K_1(\alpha)f(x)h(y)+K_0(\alpha)f'(x)h'(y)\right).
\end{aligned}
\tag{3.14}
$$

Integrating with respect to x from 0 to u

$$
\begin{aligned}
&\frac{\left[K_1(\alpha)h(y)+K_0(\alpha)h'(y)\right]}{\Gamma(1-\alpha)}\int_0^u g_2(x)(u-x)^{-\alpha}\,dx \\
&+v_1(y)\left({}_0^{CPC}D_u^{\alpha} f(u)\right) \\
&\geq \frac{v_1(y)\left(K_1(\alpha)+K_0(\alpha)\right)}{\Gamma(1-\alpha)}\int_0^u g_2(x)(u-x)^{-\alpha}\,dx \\
&+\frac{K_1(\alpha)h(y)}{\Gamma(1-\alpha)}\int_0^u f(x)(u-x)^{-\alpha}\,dx \\
&+\frac{K_0(\alpha)h'(y)}{\Gamma(1-\alpha)}\int_0^u f'(x)(u-x)^{-\alpha}\,dx.
\end{aligned}
\tag{3.15}
$$

Multiplying (3.15) by $(u-y)^{-\alpha}$ and integrating with respect to y from 0 to u

$$
\begin{aligned}
&\left({}_0^{CPC}D_u^{\alpha} f(u)\right)\int_0^u g_2(x)(u-x)^{-\alpha}\,dx \\
&+\left({}_0^{CPC}D_u^{\alpha} f(u)\right)\int_0^u v_1(y)(u-y)^{-\alpha}\,dy \\
&\geq \frac{\left(K_1(\alpha)+K_0(\alpha)\right)}{\Gamma(1-\alpha)}\int_0^u g_2(x)(u-x)^{-\alpha}\,dx\int_0^u v_1(y)(u-y)^{-\alpha}\,dy
\end{aligned}
\tag{3.16}
$$

$$+\frac{K_1(\alpha)}{\Gamma(1-\alpha)}\int_0^u f(x)(u-x)^{-\alpha}dx\int_0^u h(y)(u-y)^{-\alpha}dy$$
$$+\frac{K_0(\alpha)}{\Gamma(1-\alpha)}\int_0^u f'(x)(u-x)^{-\alpha}dx\int_0^u h'(y)(u-y)^{-\alpha}dy.$$

which completes the proof.

REFERENCES

1. H. Ahmad, M. Tariq, S.K. Sahoo, S. Askar, A.E. Abouelregal, K.M. Khedher, Refinements of ostrowski type integral inequalities involving Atangana–Baleanu fractional integral operator, Symmetry, 13(11) (2021), 2059.
2. D. Baleanu, A. Fernandez, A. Akgul, On a fractional operator combining proportional and classical differintegrals, Mathematics, 8 (2020), 360.
3. M. Bezziou, Z. Dahmani, New integral operators for conformable fractional calculus with applications, Journal of Interdisciplinary Mathematics, 2022, 1-14.
4. R. Gorenflo, F. Mainardi, Fractional calculus. In Fractals and fractional calculus in continuum mechanics (pp. 223-276). Springer, Vienna, 1997.
5. M. Gürbüz, A.O. Akdemir, M.A. Dokuyucu, Novel approaches for differentiable convex functions via the proportional Caputo-hybrid operators, Fractal and Fractional, 6(5) (2022), 258.
6. M.B. Khan, M.A. Noor, T. Abdeljawad, A.A.A. Mousa, B. Abdalla, S.M. Alghamdi, LR-preinvex interval-valued functions and Riemann–Liouville fractional integral inequalities, Fractal and Fractional, 5(4) (2021), 243.
7. J.T. Machado, V. Kiryakova, F. Mainardi, Recent history of fractional calculus. Communications in nonlinear science and numerical simulation, 16(3) (2011), 1140-1153.
8. B.F. Martnez-Salgado, R. Rosas-Sampayo, A. Torres-Hernández, C. Fuentes, Application of fractional calculus to oil industry,Fractal Anal. Appl. Phys. Eng. Technol., 2017, 21-42.
9. Y.A. Rossikhin, Reflections on two parallel ways in the progress of fractional calculus in mechanics of solids, Applied Mechanics Reviews, 63(010701) (2010), 1-12.

New Fractional Integral Inequalities for Different Types of Convex Functions

Coşkun YALÇINKAYA[1] and Mustafa GÜRBÜZ[2]

[1]*Institute of Graduate Education*
Ağrı İbrahim Çeçen University, Ağrı-TURKEY
E-mail: csknylcnky@gmail.com
[*2]*Department of Elementary Mathematics Education, Faculty of Education,*
Ağrı İbrahim Çeçen University, Ağrı-TURKEY
E-mail: mgurbuz@agri.edu.tr

ABSTRACT

In this paper we obtained some new fractional inequalities for different kinds of convex functions using the proportional Caputo-hybrid operator with fairly elementary analysis. Since the proportional Caputo-hybrid operator is important in that its special cases gives a linear combination of Riemann–Liouville integral and a Caputo derivative, it was deemed appropriate to be used in this study.

1. INTRODUCTION

Fractional calculus was first suggested for consideration by Leibnitz in his letter to L'Hospital which dealt with derivatives of order $\alpha = \frac{1}{2}$ (see [1]). Hereupon, this theory has been used in many fields of science such as economics, biology, engineering, physics and mathematics for sure. Many types of fractional derivatives and integrals were studied by Hadamard, Caputo, Riemann-Liouville, Grönwald- Letnikov, etc. Various properties of these operators have been summarized in [9]. For the last decades, this theory has been used in inequality theory frequently because it enables scientists to obtain integral inequalities for also non-integer orders. One of the most famous inequality is Ostrowski's which has lead to gain many practical inequalities with fractional calculus as well.

Fractional calculus has been appealing to many researchers over the last decades ([4], [6]).Some researchers have found that different fractional derivatives with different singular or nonsingular kernels need to be identified by real-world problems in different fields of engineering and science ([8], [7]). These different fractional operators are also used in integral inequalities ([5]). Thus, fractional calculus plays an important role in the development of inequality theory. One of the fractional operators obtained in the last years is so-called Caputo-hybrid operator is given in the following:

Definition 1 *(see [2]) Let $f : I \subseteq \mathbb{R}^+ \to \mathbb{R}$ be a differentiable function on I°. Also let f and f' are L^1 functions on I. Then, the proportional Caputo-hybrid operator may be defined as*

$$ {}_0^{CPC}D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \left(K_1(\alpha) f(x) + K_0(\alpha) f'(x)\right)(t-x)^{-\alpha} dx $$

where $\alpha \in [0,1]$ and K_0 and K_1 are functions satisfing

$$ \lim_{\alpha \to 0^+} K_0(\alpha) = 0; \qquad \lim_{\alpha \to 1^-} K_0(\alpha) = 1; \qquad K_0(\alpha) \neq 0, \quad \alpha \in (0,1]; \tag{1.1} $$

$$ \lim_{\alpha \to 0^+} K_1(\alpha) = 1; \qquad \lim_{\alpha \to 1^-} K_1(\alpha) = 0; \qquad K_1(\alpha) \neq 0, \quad \alpha \in [0,1). \tag{1.2} $$

Erdelyi et. al deeply involved in hypergeometric functions which Whittaker discovered in 1904 and gave the definition of iti in [9] as:

$$ {}_2F_1(a,b;c;z) = \frac{1}{\beta(b,b-c)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt, \;\; c > b > 0, \;\; |z| < 1. $$

In this paper some new inequalities for different kinds of convex functions are obtained by using the proportional Caputo-hybrid operator.

2. SOME RESULTS FOR DIFFERENT KINDS OF CONVEX FUNCTIONS

Theorem 1 *Let $f : I \subseteq \mathbb{R}^+ \to \mathbb{R}$ be a differentiable function on I°. Also let f and f' are L^1 functions on I. If f and f' are convex on I, then the following inequality holds*

$$ \begin{aligned} & {}_0^{CPC}D_t^{\alpha} f(u) \\ & \leq \frac{(b-a)^{\alpha-3}}{\Gamma(1-\alpha)} \left(\frac{u^{2-\alpha}}{2-\alpha} + \frac{(b-a)(b-u)u^{1-\alpha}}{1-\alpha} \right) \\ & \times \left[K_1(\alpha) f(a) + K_0(\alpha) f'(a)\right] \\ & + \frac{(b-a)^{\alpha-1}}{\Gamma(1-\alpha)} \left(\frac{u^{1-\alpha}}{1-\alpha} - \frac{u^{2-\alpha}}{(b-a)^2(2-\alpha)} - \frac{(b-u)u^{1-\alpha}}{(b-a)(1-\alpha)} \right) \\ & \times \left[K_1(\alpha) f(b) + K_0(\alpha) f'(b)\right] \end{aligned} $$

where $\alpha \in [0,1]$ with K_0 and K_1 are functions satisfing the conditions (1.1) and (1.2).

Proof. By using definition of the proportional Caputo-hybrid operator, properties of modulus and changing variables as $x = ta + (1-t)b$ we get

$$ \begin{aligned} & {}_0^{CPC}D_t^{\alpha} f(u) \\ & = \frac{1}{\Gamma(1-\alpha)} \int_0^u \left(K_1(\alpha) f(x) + K_0(\alpha) f'(x)\right)(u-x)^{-\alpha} dx \end{aligned} $$

$$
\begin{aligned}
&= \frac{b-a}{\Gamma(1-\alpha)} \int_{\frac{b-u}{b-a}}^{\frac{b}{b-a}} \begin{pmatrix} K_1(\alpha) f(ta+(1-t)b) \\ + K_0(\alpha) f'(ta+(1-t)b) \end{pmatrix} (u-(ta+(1-t)b))^{-\alpha} dt \\
&= \frac{(b-a)K_1(\alpha)}{\Gamma(1-\alpha)} \int_{\frac{b-u}{b-a}}^{\frac{b}{b-a}} f(ta+(1-t)b)(u-(ta+(1-t)b))^{-\alpha} dt \\
&+ \frac{(b-a)K_0(\alpha)}{\Gamma(1-\alpha)} \int_{\frac{b-u}{b-a}}^{\frac{b}{b-a}} f'(ta+(1-t)b)(u-(ta+(1-t)b))^{-\alpha} dt. \qquad (2.1)
\end{aligned}
$$

Using convexity of f and f' we get

$$
\begin{aligned}
&{}_0^{CPC}D_t^{\alpha} f(u) \\
&\leq \frac{(b-a)K_1(\alpha)}{\Gamma(1-\alpha)} \int_{\frac{b-u}{b-a}}^{\frac{b}{b-a}} (tf(a)+(1-t)f(b))(u-(ta+(1-t)b))^{-\alpha} dt \\
&+ \frac{(b-a)K_0(\alpha)}{\Gamma(1-\alpha)} \int_{\frac{b-u}{b-a}}^{\frac{b}{b-a}} (tf'(a)+(1-t)f'(b))(u-(ta+(1-t)b))^{-\alpha} dt.
\end{aligned}
$$

By simple calculation we have

$$
\begin{aligned}
&{}_0^{CPC}D_t^{\alpha} f(u) \\
&\leq \frac{(b-a)K_1(\alpha)}{\Gamma(1-\alpha)} \Bigg[f(a)\left(\frac{1}{(b-a)^{4-\alpha}}\left(\frac{u^{2-\alpha}}{2-\alpha} + \frac{(b-a)(b-u)u^{1-\alpha}}{1-\alpha}\right)\right) \\
&+ f(b)\left(\frac{u^{1-\alpha}}{(b-a)^{2-\alpha}(1-\alpha)} - \frac{u^{2-\alpha}}{(b-a)^{4-\alpha}(2-\alpha)} - \frac{(b-u)u^{1-\alpha}}{(b-a)^{3-\alpha}(1-\alpha)}\right)\Bigg] \\
&+ \frac{(b-a)K_0(\alpha)}{\Gamma(1-\alpha)} \Bigg[f'(a)\left(\frac{1}{(b-a)^{4-\alpha}}\left(\frac{u^{2-\alpha}}{2-\alpha} + \frac{(b-a)(b-u)u^{1-\alpha}}{1-\alpha}\right)\right) \\
&+ f'(b)\left(\frac{u^{1-\alpha}}{(b-a)^{2-\alpha}(1-\alpha)} - \frac{u^{2-\alpha}}{(b-a)^{4-\alpha}(2-\alpha)} - \frac{(b-u)u^{1-\alpha}}{(b-a)^{3-\alpha}(1-\alpha)}\right)\Bigg]
\end{aligned}
$$

which is the desired result.

Theorem 2 *Let $f : I \subseteq \mathbb{R}^+ \to \mathbb{R}$ be a differentiable function on I°. Also let f and f' are L^1 functions on I. If f and f' are $s-$convex (in the second sense) on I, then the following inequality holds*

$$
\begin{aligned}
&{}_0^{CPC}D_t^{\alpha} f(u) \\
&\leq \frac{(b-u)^s u^{1-\alpha}}{\Gamma(2-\alpha)(b-a)^{2s-\alpha}} \left({}_2F_1\left[-s, 1-\alpha; 2-\alpha; \frac{-u}{(b-a)(b-u)}\right]\right)
\end{aligned}
$$

$$\times\left[K_1(\alpha)f(a)+K_0(\alpha)f'(a)\right]$$
$$+\frac{(u-a)^s u^{1-\alpha}}{\Gamma(2-\alpha)(b-a)^{2s-\alpha}}\left({}_2F_1\left[-s,1-\alpha;2-\alpha;\frac{u}{(b-a)(u-a)}\right]\right)$$
$$\times\left[K_1(\alpha)f(b)+K_0(\alpha)f'(b)\right]$$

where $s\in(0,1]$, $\alpha\in[0,1]$ with K_0 and K_1 are functions satisfing the conditions (1.1) and (1.2).

Proof. Using 2.1 and $s-$convexity of f and f' we get

$$\begin{aligned}
&{}_0^{CPC}D_t^{\alpha} f(u)\\
&=\frac{(b-a)K_1(\alpha)}{\Gamma(1-\alpha)}\int_{\frac{b-u}{b-a}}^{\frac{b}{b-a}} f(ta+(1-t)b)(u-(ta+(1-t)b))^{-\alpha}\,dt\\
&+\frac{(b-a)K_0(\alpha)}{\Gamma(1-\alpha)}\int_{\frac{b-u}{b-a}}^{\frac{b}{b-a}} f'(ta+(1-t)b)(u-(ta+(1-t)b))^{-\alpha}\,dt.\\
&\le\frac{(b-a)K_1(\alpha)}{\Gamma(1-\alpha)}\int_{\frac{b-u}{b-a}}^{\frac{b}{b-a}} \left(t^s f(a)+(1-t)^s f(b)\right)(u-(ta+(1-t)b))^{-\alpha}\,dt\\
&+\frac{(b-a)K_0(\alpha)}{\Gamma(1-\alpha)}\int_{\frac{b-u}{b-a}}^{\frac{b}{b-a}} \left(t^s f'(a)+(1-t)^s f'(b)\right)(u-(ta+(1-t)b))^{-\alpha}\,dt.
\end{aligned}$$

By simple calculation we get

$$\begin{aligned}
&{}_0^{CPC}D_t^{\alpha} f(u)\\
&\le\frac{(b-a)K_1(\alpha)}{\Gamma(1-\alpha)}\left[\left(f(a)\frac{(b-u)^s u^{1-\alpha}}{(b-a)^{2s-\alpha+1}(1-\alpha)}\left({}_2F_1\left[-s,1-\alpha;2-\alpha;\frac{-u}{(b-a)(b-u)}\right]\right)\right)\right.\\
&\left.+f(b)\frac{(u-a)^s u^{1-\alpha}}{(b-a)^{2s-\alpha+1}(1-\alpha)}\left({}_2F_1\left[-s,1-\alpha;2-\alpha;\frac{u}{(b-a)(u-a)}\right]\right)\right]\\
&+\frac{(b-a)K_0(\alpha)}{\Gamma(1-\alpha)}\left[\left(f'(a)\frac{(b-u)^s u^{1-\alpha}}{(b-a)^{2s-\alpha+1}(1-\alpha)}\left({}_2F_1\left[-s,1-\alpha;2-\alpha;\frac{-u}{(b-a)(b-u)}\right]\right)\right)\right.\\
&\left.+f'(b)\frac{(u-a)^s u^{1-\alpha}}{(b-a)^{2s-\alpha+1}(1-\alpha)}\left({}_2F_1\left[-s,1-\alpha;2-\alpha;\frac{u}{(b-a)(u-a)}\right]\right)\right].
\end{aligned}$$

Theorem 3 *Let $f:I\subseteq\mathbb{R}^+\to\mathbb{R}$ be a differentiable function on I°. Also let f and f' are L^1 functions on I. If f and f' are $m-$convex on I, then the following inequality holds*

$${}_0^{CPC}D_t^{\alpha} f(u)$$

$$\begin{aligned}
&\leq \frac{(b-a)^{\alpha-3}}{\Gamma(1-\alpha)}\left(\frac{u^{2-\alpha}}{2-\alpha}+\frac{(b-a)(b-u)u^{1-\alpha}}{1-\alpha}\right)\\
&\times\left[K_1(\alpha)f(a)+K_0(\alpha)f'(a)\right]\\
&+\frac{(b-a)^{\alpha-1}}{\Gamma(1-\alpha)}\left(\frac{u^{1-\alpha}}{1-\alpha}-\frac{u^{2-\alpha}}{(b-a)^2(2-\alpha)}-\frac{(b-u)u^{1-\alpha}}{(b-a)(1-\alpha)}\right)\\
&\times\left[K_1(\alpha)mf\left(\frac{b}{m}\right)+K_0(\alpha)mf'\left(\frac{b}{m}\right)\right]
\end{aligned}$$

where $(\alpha,m)\in[0,1]^2$ with K_0 and K_1 are functions satisfing the conditions (1.1) and (1.2).
Proof. Using 2.1 and $m-$convexity of f and f' we get

$$\begin{aligned}
&{}_0^{CPC}D_t^{\alpha} f(u)\\
&=\frac{(b-a)K_1(\alpha)}{\Gamma(1-\alpha)}\int_{\frac{b-u}{b-a}}^{\frac{b}{b-a}} f(ta+(1-t)b)(u-(ta+(1-t)b))^{-\alpha}\,dt\\
&+\frac{(b-a)K_0(\alpha)}{\Gamma(1-\alpha)}\int_{\frac{b-u}{b-a}}^{\frac{b}{b-a}} f'(ta+(1-t)b)(u-(ta+(1-t)b))^{-\alpha}\,dt.\\
&\leq\frac{(b-a)K_1(\alpha)}{\Gamma(1-\alpha)}\int_{\frac{b-u}{b-a}}^{\frac{b}{b-a}}\left(tf(a)+m(1-t)f\left(\frac{b}{m}\right)\right)(u-(ta+(1-t)b))^{-\alpha}\,dt\\
&+\frac{(b-a)K_0(\alpha)}{\Gamma(1-\alpha)}\int_{\frac{b-u}{b-a}}^{\frac{b}{b-a}}\left(tf'(a)+m(1-t)f'\left(\frac{b}{m}\right)\right)(u-(ta+(1-t)b))^{-\alpha}\,dt.
\end{aligned}$$

By simple calculation we have

$$\begin{aligned}
&{}_0^{CPC}D_t^{\alpha} f(u)\\
&\leq\frac{(b-a)K_1(\alpha)}{\Gamma(1-\alpha)}\left[f(a)\left(\frac{1}{(b-a)^{4-\alpha}}\left(\frac{u^{2-\alpha}}{2-\alpha}+\frac{(b-a)(b-u)u^{1-\alpha}}{1-\alpha}\right)\right)\right.\\
&\left.+mf\left(\frac{b}{m}\right)\left(\frac{u^{1-\alpha}}{(b-a)^{2-\alpha}(1-\alpha)}-\frac{u^{2-\alpha}}{(b-a)^{4-\alpha}(2-\alpha)}-\frac{(b-u)u^{1-\alpha}}{(b-a)^{3-\alpha}(1-\alpha)}\right)\right]\\
&+\frac{(b-a)K_0(\alpha)}{\Gamma(1-\alpha)}\left[f'(a)\left(\frac{1}{(b-a)^{4-\alpha}}\left(\frac{u^{2-\alpha}}{2-\alpha}+\frac{(b-a)(b-u)u^{1-\alpha}}{1-\alpha}\right)\right)\right.\\
&\left.+mf'\left(\frac{b}{m}\right)\left(\frac{u^{1-\alpha}}{(b-a)^{2-\alpha}(1-\alpha)}-\frac{u^{2-\alpha}}{(b-a)^{4-\alpha}(2-\alpha)}-\frac{(b-u)u^{1-\alpha}}{(b-a)^{3-\alpha}(1-\alpha)}\right)\right]
\end{aligned}$$

REFERENCES

1. G.W. Leibniz, “ Letter from Hanover, Germany to G.F.A. L’Hospital, September 30, 1695” , Leibniz Mathematische Schriften. Olms-Verlag, Hildesheim, Germany, 1962. p.301–302, First published in 1849.
2. D. Baleanu, A. Fernandez, A. Akgul, On a fractional operator combining proportional and classical differintegrals, Mathematics, 8 (2020), 360.
3. M. Bezziou, Z. Dahmani, New integral operators for conformable fractional calculus with applications, Journal of Interdisciplinary Mathematics, 2022, 1-14.
4. R. Gorenflo, F. Mainardi, Fractional calculus. In Fractals and fractional calculus in continuum mechanics (pp. 223-276). Springer, Vienna, 1997.
5. M.B. Khan, M.A. Noor, T. Abdeljawad, A.A.A. Mousa, B. Abdalla, S.M. Alghamdi, LR-preinvex interval-valued functions and Riemann–Liouville fractional integral inequalities, Fractal and Fractional, 5(4) (2021), 243.
6. J.T. Machado, V. Kiryakova, F. Mainardi, Recent history of fractional calculus. Communications in nonlinear science and numerical simulation, 16(3) (2011), 1140-1153.
7. B.F. Martnez-Salgado, R. Rosas-Sampayo, A. Torres-Hernández, C. Fuentes, Application of fractional calculus to oil industry,Fractal Anal. Appl. Phys. Eng. Technol., 2017, 21-42.
8. Y.A. Rossikhin, Reflections on two parallel ways in the progress of fractional calculus in mechanics of solids, Applied Mechanics Reviews, 63(010701) (2010), 1-12.
9. W. Erdelyi, F. Magnus, Oberhettinger, et al. Higher transcendental functions, Vol. I-III, Krieger Pub., Melbourne, Florida, 1981.